33 $f(x)=x^3$, $3x-y+1=0 \Rightarrow y=3x+1$ ~~17~~

~~19~~

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x^2+2x(\Delta x)+(\Delta x)^2)(x+\Delta x)-x^3}{\Delta x}$$ ~~21~~

~~31~~

$$=\lim_{\Delta x\to 0}\frac{x^3+2x^2(\Delta x)+x(\Delta x)^2+x^2(\Delta x)+2x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^3+3x^2(\Delta x)+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}$$ 33 35

$$=\lim_{\Delta x\to 0}\frac{3x^2(\Delta x)+3x(\Delta x)^2+(\Delta x)^3}{\Delta x}=\lim_{\Delta x\to 0}3x^2+3x(\Delta x)+(\Delta x)^2=3x^2$$ 77 78

$3x^2=3$

$x^2=1 \Rightarrow x=\pm 1 \Rightarrow (1,1), (-1,-1)$

$$\boxed{y-1=3(x-1) \quad \& \quad y+1=3(x+1)}$$

(35) $f(x)=\frac{1}{\sqrt{x}}$, $x+2y-6=0 \Rightarrow 2y=-x+6 \Rightarrow y=-\frac{1}{2}x+3$

$$f'(x)=\lim_{\Delta x\to 0}\frac{\frac{1}{\sqrt{x+\Delta x}}-\frac{1}{\sqrt{x}}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{\sqrt{x}-\sqrt{x+\Delta x}}{\sqrt{x+\Delta x}\cdot\sqrt{x}}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{x}-\sqrt{x+\Delta x}}{\Delta x\sqrt{x+\Delta x}\sqrt{x}}\cdot\frac{\sqrt{x}+\sqrt{x+\Delta x}}{\sqrt{x}+\sqrt{x+\Delta x}}$$

$$=\lim_{\Delta x\to 0}\frac{x-(x+\Delta x)}{\text{STUFF}}=\lim_{\Delta x\to 0}\frac{-\Delta x}{\text{STUFF}}=\lim_{\Delta x\to 0}\frac{-1}{\sqrt{x+\Delta x}\sqrt{x}(\sqrt{x}+\sqrt{x+\Delta x})}=\frac{-1}{\sqrt{x}\cdot\sqrt{x}(\sqrt{x}+\sqrt{x})}$$

$$=\frac{-1}{x(2\sqrt{x})}=\frac{-1}{2x\sqrt{x}}$$

$$\frac{-1}{2x\sqrt{x}}=-\frac{1}{2} \Rightarrow x=1 \Rightarrow (1,1)$$

$$\boxed{y-1=-\frac{1}{2}(x-1)}$$

(77) $f(x)=(x-6)^{2/3}$, $c=6$

$$f'(x)=\lim_{x\to 6}\frac{f(x)-f(6)}{x-6}=\lim_{x\to 6}\frac{(x-6)^{2/3}-0}{(x-6)}=\lim_{x\to 6}\frac{(x-6)^{2/3}}{(x-6)}=\lim_{x\to 6}\frac{1}{(x-6)^{1/3}}\Rightarrow \text{limit DNE}$$

(78) $f(x)=(x+3)^{1/3}$, $c=-3$

$$f'(x)=\lim_{x\to -3}\frac{(x+3)^{1/3}-0}{x+3}=\lim_{x\to -3}\frac{(x+3)^{1/3}}{x+3}=\lim_{x\to -3}\frac{1}{(x+3)^{2/3}}\Rightarrow \text{limit DNE}$$

22) $$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{x^2 - (x+\Delta x)^2}{x^2(x+\Delta x)^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 - (x+\Delta x)^2}{(\Delta x)(x^2)(x+\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (x^2 + 2x(\Delta x) + (\Delta x)^2)}{(\Delta x)(x^2)(x+\Delta x)^2} = \lim_{\Delta x \to 0} \frac{-2x(\Delta x) - (\Delta x)^2}{(\Delta x)(x^2)(x+\Delta x)^2} = \lim_{\Delta x \to 0} \frac{-2x - (\Delta x)}{x^2(x+\Delta x)^2}$$

$$= \frac{-2x}{x^2(x^2)} = \frac{-2x}{x^4} = \boxed{\frac{-2}{x^3}}$$

24) $$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{4}{\sqrt{x+\Delta x}} - \frac{4}{\sqrt{x}}}{\Delta x} = \lim_{\Delta x \to 0} \frac{4\sqrt{x} - 4\sqrt{x+\Delta x}}{(\Delta x)\sqrt{x}\,\sqrt{x+\Delta x}} \cdot \frac{4\sqrt{x} + 4\sqrt{x+\Delta x}}{4\sqrt{x} + 4\sqrt{x+\Delta x}} = \lim_{\Delta x \to 0} \frac{16x - 16(x+\Delta x)}{(\Delta x)(\sqrt{x})(\sqrt{x+\Delta x})(4\sqrt{x}+4\sqrt{x+\Delta x})}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{16x} - \cancel{16x} - 16(\Delta x)}{\text{STUFF}} = \lim_{\Delta x \to 0} \frac{-16}{\text{STUFF}} = \frac{-16}{\sqrt{x}\cdot\sqrt{x}\,(4\sqrt{x}+4\sqrt{x})} = \frac{-16}{x(8\sqrt{x})} = \frac{-16}{8x\sqrt{x}} = \boxed{\frac{-2}{x\sqrt{x}}}$$

Situations Where Derivatives Do Not Exist

1) Corners → 1-sided derivatives differ
   ↳ absolute value functions

2) Cusp → slopes of secant lines approach ∞ from one side & $-\infty$ from other (extreme corner)
   ↳ <u>EX</u> → $f(x) = x^{2/3}$

3) Vertical tangent → slopes of secant lines approach $\pm\infty$ from both sides
   ↳ <u>EX</u> → $f(x) = \sqrt[3]{x}$

4) Discontinuity → one or both of one-sided limits don't exist
   ↳ <u>EX</u> → step functions

- "locally linear" → function that is differentiable @ $x=a$ closely resembles own tangent line close to $x=a$

→ differentiability implies continuity
  ↳ if $f(x)$ has a derivative @ $x=a$, then it is continuous @ $x=a$

→ Intermediate Value Theorem for Derivatives
  ↳ If a + b are any 2 pts in an interval on which $f(x)$ is differentiable, then $f'(x)$ takes on every value b/w $f'(a)$ + $f'(b)$

- Basic Differentiation Rules
  - Constant ⇒ If a is a real #, then $\frac{d}{dx}[a] = 0$
  - Power Rule ⇒ If n is a rational #, then x^n is differentiable & $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$

EX1 → $f(x) = x^4 \Rightarrow f'(x) = 4 \cdot x^{4-1} = \boxed{4x^3}$

EX2 → $f(x) = 3x^2 \Rightarrow f'(x) = 2 \cdot 3 \cdot x^{2-1} = \boxed{6x}$

EX3 → $f(x) = \frac{4}{x^3} = 4x^{-3} \Rightarrow f'(x) = \boxed{-12x^{-4}}$

EX4 → $f(x) = 5\sqrt[3]{x} = 5x^{1/3} \Rightarrow f'(x) = \boxed{\frac{5}{3}x^{-2/3}}$

HW: p. 115 → 3-18, 39-44

- Sum/Difference Rule $\Rightarrow$ If $f(x)$ & $g(x)$ are differentiable functions, then their sum/difference is differentiable everywhere $f(x)$ & $g(x)$ are differentiable

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right] = f'(x) \pm g'(x)$$

- Product Rule $\Rightarrow$ If $f(x)$ & $g(x)$ are differentiable functions, then their product is also differentiable where $f(x)$ & $g(x)$ are differentiable

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

EX1 → $y=(x^2+2)(4x^3-6x^2+7x-5)$

$$y'=(x^2+2)(12x^2-12x+7)+(4x^3-6x^2+7x-5)(2x)$$
$$=12x^4-12x^3+7x^2+24x^2-24x+14+8x^4-12x^3+14x^2-10x$$
$$=20x^4-24x^3+45x^2-34x+14$$

EX2 → $y=(4x^6-10x^5+8)(2x-\frac{2}{x})$ → $-2x^{-1}$

$$y'=(4x^6-10x^5+8)(2+2x^{-2})+(2x-\tfrac{2}{x})(24x^5-50x^4)$$

⋮

EX3 → $y=(3x^4+7x)(5x^3+\frac{5}{x^2})$ → $5x^{-2}$

$$y'=(3x^4+7x)(15x^2-10x^{-3})+(5x^3+\tfrac{5}{x^2})(12x^3+7)$$

⋮

- Quotient Rule ⇒ If $f(x)$ & $g(x)$ are differentiable functions, then the quotient is also differentiable where $f(x)$ & $g(x)$ are differentiable <u>AND</u> where $g(x) \neq 0$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

<u>EX1</u> → $y = \dfrac{x^2+1}{5x+7}$

$$y' = \frac{(5x+7)(2x) - (x^2+1)(5)}{(5x+7)^2} = \frac{10x^2 + 14x - 5x^2 - 5}{(5x+7)^2} = \frac{5x^2+14x-5}{(5x+7)^2}$$

<u>EX2</u> → $y = \dfrac{3x^2+2x}{6x-4}$

$$y' = \frac{(6x-4)(6x+2) - (3x^2+2x)(6)}{(6x-4)^2} = \frac{36x^2 - 12x - 8 - 18x^2 - 12x}{(6x-4)^2} = \frac{18x^2 - 24x - 8}{(6x-4)^2}$$

HW: p. 126 → 1-4, 7-10, 13-16,
25-37 odd

9) $f(x) = \dfrac{\sqrt[3]{x}}{x^3+1} = \dfrac{x^{1/3}}{x^3+1}$

$$f'(x) = \frac{(x^3+1)(\frac{1}{3}x^{-2/3}) - (x^{1/3})(3x^2)}{(x^3+1)^2} = \frac{(\frac{1}{3}x^{7/3} + \frac{1}{3}x^{-2/3}) - 3x^{7/3}}{(x^3+1)^2} = \frac{-\frac{8}{3}x^{7/3} + \frac{1}{3}x^{-2/3}}{(x^3+1)^2}$$

25
27
29 33
37

25) $f(x) = \dfrac{3-2x-x^2}{x^2-1}$

$$f'(x) = \frac{(x^2-1)(-2-2x) - (3-2x-x^2)(2x)}{(x^2-1)^2} = \frac{-2x^2+2-\cancel{2x^3}+2x-6x+4x^2+\cancel{2x^3}}{(x^2-1)^2} = \frac{2x^2-4x+2}{(x^2-1)^2}$$

$$= \frac{2(x^2-2x+1)}{(x^2-1)^2} = \frac{2(x-1)^2}{(x+1)^2(x-1)^2} = \boxed{\frac{2}{(x+1)^2}}$$

27) $f(x) = x\left(1-\dfrac{4}{x+3}\right) = x\left(\dfrac{x+3-4}{x+3}\right) = x\left(\dfrac{x-1}{x+3}\right) = \dfrac{x^2-x}{x+3}$

$$f'(x) = \frac{(x+3)(2x-1)-(x^2-x)(1)}{(x+3)^2} = \frac{2x^2+6x-x-3-x^2+x}{(x+3)^2} = \frac{x^2+6x-3}{(x+3)^2}$$

(29) $f(x) = \frac{2x+5}{\sqrt{x}} = \frac{2x+5}{x^{1/2}}$

$f'(x) = \frac{(x^{1/2})(2) - (2x+5)(\frac{1}{2}x^{-1/2})}{(x^{1/2})^2} = \frac{2x^{1/2} - x^{1/2} - \frac{5}{2}x^{-1/2}}{x} = \frac{x^{1/2} - \frac{5}{2}x^{-1/2}}{x}$

(33) $f(x) = \frac{2-\frac{1}{x}}{x-3} = \frac{\frac{2x-1}{x}}{\frac{x-3}{1}} = \frac{2x-1}{x(x-3)} = \frac{2x-1}{x^2-3x}$

$f'(x) = \frac{(x^2-3x)(2) - (2x-1)(2x-3)}{(x^2-3x)^2} = \frac{2x^2-6x-(4x^2-2x-6x+3)}{(x^2-3x)^2} = \frac{-2x^2+2x-3}{(x^2-3x)^2}$

(35) $f(x) = (3x^3+4x)(x-5)(x+1) = (3x^3+4x)(x^2-4x-5)$

$f'(x) = (3x^3+4x)(2x-4) + (x^2-4x-5)(9x^2+4) = 6x^4+8x^2-12x^3-16x+9x^4-36x^3-45x^2+4x^2-16x-20$

$= 15x^4-48x^3-33x^2-32x-20$

(37) $f(x) = \frac{x^2+c^2}{x^2-c^2}$

$f'(x) = \frac{(x^2-c^2)(2x) - (x^2+c^2)(2x)}{(x^2-c^2)^2} = \frac{2x^3-2c^2x-2x^3-2c^2x}{(x^2-c^2)^2} = \boxed{\frac{-4c^2x}{(x^2-c^2)^2}}$