Differential Equations

- A differential equation of the form $\frac{dy}{dx} = f(y) \cdot g(x)$ is called <u>separable</u>.
  - → We separate variables by writing $\frac{1}{f(y)}\,dy = g(x)\,dx$
  - → The solution is found by integrating each side w/ respect to its isolated variable

<u>EX1</u> → Solve for y if $\frac{dy}{dx} = x^2y^2$, $(0,3)$

$$\int \frac{1}{y^2}\,dy = \int x^2\,dx$$

$$\frac{-1}{y} = \frac{1}{3}x^3 + C$$ ← general solution

$$\frac{-1}{3} = \cancel{\frac{1}{3}(0)^3}^{\,0} + C$$

$$-\frac{1}{3} = C$$

$$\frac{-1}{y} = \frac{1}{3}x^3 - \frac{1}{3}$$ → specific solution

$$-\frac{1}{y} = \frac{x^3-1}{3}$$

$$\frac{1}{y} = \frac{-x^3+1}{3}$$

$$\boxed{y = \frac{3}{-x^3+1}}$$

EX2 → Solve for y if $\frac{dy}{dx} = \frac{-x}{y}$, $(4,3)$

$$\int y\,dy = \int -x\,dx$$

$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ ← general solution

$$\frac{1}{2}(3)^2 = -\frac{1}{2}(4)^2 + C$$

$$\frac{9}{2} = -8 + C$$

$\frac{25}{2} = C \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + \frac{25}{2}$ ← specific solution

$$y^2 = -x^2 + 25$$

$y = \sqrt{-x^2+25}$, $(-5,5)$

EX3 → Solve for y if $\frac{dy}{dx} = \frac{y}{x}$, (1, 1)

$$\frac{dy}{dx} = y \cdot \frac{1}{x}$$

$$dy = y \cdot \frac{1}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + C$$

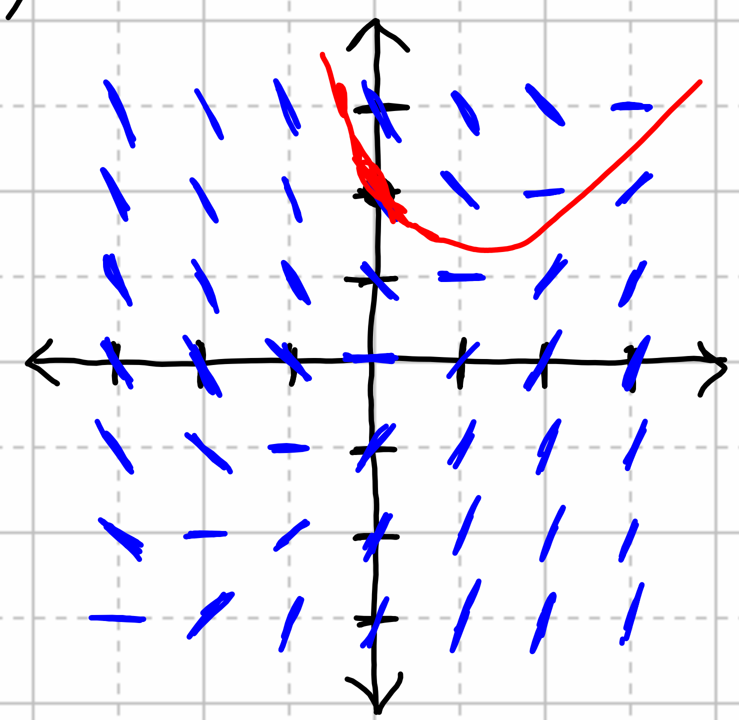
$$\cancel{\ln|1|}^{\,0} = \cancel{\ln|1|}^{\,0} + C$$

$$0 = C \Rightarrow \ln|y| = \ln|x|$$

$$\boxed{y = x}$$

- Slope fields → short line segments displaying slopes of solution curves through those points

EX → Sketch slope field for the differential equation $y' = x - y$, and sketch a solution through $(0,2)$



- In many instances, the rate of change of a variable y is proportional to the value of y. If y is a function of time t, the proportion can be written as

$$\frac{dy}{dt} = ky \quad \text{(Rate of change of } y \text{ is proportional to } y\text{)}$$

$$dy = k \cdot y \cdot dt$$

$$\int \frac{1}{y}\, dy = \int k \cdot dt$$

$$\ln|y| = kt + C$$

$$y = e^{kt+C} = e^{kt} \cdot e^{C}$$

$$y = C_1 e^{kt} \quad (C_1 = \text{Initial value of } y \ (t=0))$$

EX1 → Rate of change of y is proportional to y

$t=0,\ y=2 \Rightarrow y = 2e^{kt}$

$t=2,\ y=4$

$t=3,\ y=?$

1) Solve for k

$4 = 2e^{k(2)}$

$2 = e^{2k}$

$\ln 2 = 2k$

$k = \frac{\ln 2}{2} \Rightarrow y = 2e^{\frac{\ln 2}{2}t}$

2) Solve for y

$y = 2e^{\frac{\ln 2}{2}(3)}$

$y = 5.657$

EX2 → Rate of change of N is proportional to N. When $t=0$, $N=250$, and when $t=1$, $N=400$. What is the value of N when $t=4$?

$$N = 250e^{kt}$$

$$400 = 250e^{k(1)}$$

$$\frac{8}{5} = e^{k}$$

$$k = \ln\left(\frac{8}{5}\right) \Rightarrow N = 250e^{\ln\left(\frac{8}{5}\right)t}$$

$$N = 250e^{\ln\left(\frac{8}{5}\right)\cdot 4}$$

$$\boxed{N = 1638.400}$$

HW: p. 418 → 21-28, 62

(23) $V = 20000e^{kt}$

$12500 = 20000e^{k(4)}$

$\frac{5}{8} = e^{4k}$

$\ln\left(\frac{5}{8}\right) = 4k$

$k = \frac{\ln\left(\frac{5}{8}\right)}{4}$

$V = 20000e^{\frac{\ln\left(\frac{5}{8}\right)}{4}\cdot 6}$

$\boxed{V = 9882.118}$

(27) $y = Ce^{kt}$

$\frac{y}{e^{kt}} = C$

$\frac{1}{e^{k}} = C$ $\qquad \frac{5}{e^{5k}} = C$

$\frac{1}{e^{k}} = \frac{5}{e^{5k}}$

$e^{5k} = 5e^{k}$

$e^{4k} = 5$

$4k = \ln 5$

$k = \frac{\ln 5}{4}$

$1 = Ce^{\frac{\ln 5}{4}\cdot(1)}$

$C = 0.669$

$\downarrow$

$\boxed{y = 0.669e^{\frac{\ln 5}{4}t}}$

28

62

28

$y = Ce^{kt}$

$\frac{y}{e^{kt}} = C$

$\frac{\frac{1}{2}}{e^{3k}} = C \qquad \frac{5}{e^{4k}} = C$

$\frac{\frac{1}{2}}{e^{3k}} = \frac{5}{e^{4k}}$

$\frac{1}{2}e^{4k} = 5e^{3k}$

$e^{k} = 10$

$k = \ln 10$

$5 = Ce^{\ln 10 \cdot 4}$

$C = \frac{5}{e^{4\ln 10}}$

$C = 0.0005$

$\downarrow$

$\boxed{y = 0.0005e^{\ln 10 \cdot t}}$

EX3 → Newton's Law of Cooling (Ex. 6, p. 417)

$$\frac{dT}{dt} = k(T - T_c)$$

↑ constant temp.

$$\frac{dT}{dt} = k(T-60)$$

$$dT = k(T-60)\,dt$$

$$\int \frac{1}{T-60}\,dT = \int k\,dt$$

$$\ln|T-60| = kt + C$$

$$T - 60 = e^{kt+C} = e^{kt} \cdot e^{C} = C_1 e^{kt}$$

$$T = C_1 e^{kt} + 60$$

$t=0$, $T=100°$

$t=10$, $T=90°$

$t=?$, $T=80°$

$$100 = C_1 \cancel{e^{k(0)}}^{1} + 60$$

$$100 = C_1 + 60$$

$$C_1 = 40$$

$$90 = 40e^{k(10)} + 60$$

$$30 = 40e^{10k}$$

$$\frac{3}{4} = e^{10k}$$

$$k = \frac{\ln\left(\frac{3}{4}\right)}{10}$$

$$80 = 40e^{\frac{\ln\left(\frac{3}{4}\right)}{10}t} + 60$$

$$20 = 40e^{\frac{\ln\left(\frac{3}{4}\right)}{10}t}$$

$$\frac{1}{2} = e^{\frac{\ln\left(\frac{3}{4}\right)}{10}t}$$

$$\ln\left(\frac{1}{2}\right) = \frac{\ln\left(\frac{3}{4}\right)}{10}t$$

$$t = \frac{10\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{3}{4}\right)} \approx \boxed{24.094}$$

- Interpreting Verbal Statements
  - Rate of Change is:
    - → Proportional ⇒ $\frac{dx}{dt} = kx$
    - → Inversely Proportional ⇒ $\frac{dx}{dt} = \frac{k}{x}$
    - → Varies Directly (Jointly) ⇒ $\frac{dx}{dt} = kxt$

<u>EX1</u> → Rate of change of y w/ respect to x is inversely proportional to the cube root of x

$$\frac{dy}{dx} = \frac{k}{x^{1/3}}$$

$$\int dy = \int \frac{k}{x^{1/3}}\,dx = \int kx^{-1/3}\,dx$$

$$\boxed{y = \frac{3}{2}kx^{2/3} + C}$$

EX2 → Rate of change of y w/ respect to x varies directly to x & y+25

$$\frac{dy}{dx} = kx(y+25)$$

$$\int \frac{1}{y+25}\,dy = \int kx\,dx$$

$$\ln|y+25| = \frac{1}{2}kx^2 + C$$

$$y+25 = e^{\frac{1}{2}kx^2} \cdot e^{C}$$

$$\boxed{y = C_1 e^{\frac{1}{2}kx^2} - 25}$$

HW: p. 418 → 1-14, 41-55 odd, 70-72

EX → "Time shift" (Ex. 4, p. 416)

$t=2,\ F=100 \Rightarrow t=0,\ F=100$

$t=4,\ F=300 \Rightarrow t=2,\ F=300$

$t=0,\ F=? \Rightarrow t=-2,\ F=?$

$F=100e^{kt}$

$300=100e^{k(2)}$

$3=e^{2k}$

$\ln 3=2k$

$k=\frac{\ln 3}{2} \Rightarrow F=100e^{\frac{\ln 3}{2}t}$

$F=100e^{\frac{\ln 3}{2}(-2)}$

$\boxed{F=33.333 \text{ flies}}$

<u>HW</u>: p. 409 → 12-48 mult. 4

p. 429 → 13-23 odd