- To approximate the value of a function at a given x-value, we often use the tangent line approximation of a nearby point at $(c, f(c))$. This can be written as:

$$y - f(c) = f'(c)(x-c)$$
$$y = f'(c)(x-c) + f(c)$$

EX → Write the tangent line of $y = 1 + \cos x$ at $\left(\frac{\pi}{2}, 1\right)$ and use it to approximate $x = \frac{3\pi}{5}$

$f'(x) = -\sin x$

$f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$

$y - 1 = -1\left(x - \frac{\pi}{2}\right)$

$y - 1 = -1\left(\frac{3\pi}{5} - \frac{\pi}{2}\right)$

$y - 1 = -1\left(\frac{6\pi}{10} - \frac{5\pi}{10}\right)$

$y - 1 = -1\left(\frac{\pi}{10}\right)$

$\boxed{y = -\frac{\pi}{10} + 1}$

NOTE: The closer the x-values are together, the better the approximation

- Let $y=f(x)$ be a differentiable function on an open interval containing x. The differential of x (dx) is any non-zero real number. The differential of y (dy) is $dy=f'(x)\,dx$

NOTE 1: Remember, the actual change in y is denoted as $\Delta y=f(c+\Delta x)-f(c)$ and can be approximated as $\Delta y \approx f'(x)\cdot\Delta x$ (from tangent line equation)

NOTE 2: Often, the differential of y can be used as an approximation of the change of y. Thus, it could be expressed as $\Delta y \approx dy$ OR $\Delta y \approx f'(x)\,dx$

EX → Let $y=x^3$. Find dy when $x=2$ and $dx=0.01$. Compare this value w/ Δy for $x=2$ + $\Delta x=0.01$

$$y'=3x^2$$

$$dy=\overset{f'(x)}{f'(2)}\cdot\overset{dx}{(0.01)}=12\cdot 0.01=0.12$$

$$\Delta y=f(2.01)-f(2)=8.120601-8=0.120601$$

HW: p. 240 → 1–19 odd

7) $y = \frac{1}{2}x^3$

$y' = \frac{3}{2}x^2$

$dy = \frac{3}{2}(2)^2(0.1) = 0.6$

$\Delta y = \frac{1}{2}(2.1)^3 - \frac{1}{2}(2)^3 = 0.6305$

9) $y = x^4 + 1$

$y' = 4x^3$

$dy = 4(-1)^3(0.01) = -0.04$

$\Delta y = (-0.99)^4 \cancel{+1} - ((-1)^4 \cancel{+1}) = -0.039$

17) $y = 2x - \cot^2 x = 2x - (\cot x)^2$

$\cancel{dx} \cdot \frac{dy}{\cancel{dx}} = 2 - 2\cot x \cdot -\csc^2 x = 2 + 2\cot x \csc^2 x \cdot dx$

$$\boxed{dy = (2 + 2\cot x \csc^2 x)dx}$$

19) $y = \frac{1}{3}\cos\left(\frac{6\pi x - 1}{2}\right) = \frac{1}{3}\cos\left(3\pi x - \frac{1}{2}\right)$

$\frac{dy}{dx} = \frac{1}{\cancel{3}} \cdot -\sin\left(3\pi x - \frac{1}{2}\right) \cdot \cancel{3}\pi$

$\cancel{dx} \cdot \frac{dy}{\cancel{dx}} = -\pi \sin\left(3\pi x - \frac{1}{2}\right) \cdot dx$

$$\boxed{dy = -\pi \sin\left(3\pi x - \frac{1}{2}\right)dx}$$

- Differentials can also be used to determine error in measurements

$$\underbrace{dy}_{\text{Propagated Error}} \approx \Delta y = \underbrace{f(x + \underbrace{\Delta x}_{\text{Measurement Error}})}_{\text{Exact Value}} - \underbrace{f(x)}_{\text{Measured Value}}$$

$$\text{Relative Error} = \frac{dy}{y}$$

EX1 → The measurement of the edge of a cube is 18 in., w/ a possible error of 0.03 in. Approximate the maximum possible propagated error in volume & surface area.

$V = s^3$

$dV = 3s^2 \cdot ds$

$dV = 3(18 \text{ in})^2(0.03 \text{ in.}) = \pm 29.16 \text{ in}^3$

$\text{Rel. Err.} = \dfrac{29.16}{(18)^3} = 0.005 \Rightarrow 0.5\%$

$SA = 6s^2$

$dA = 12s \cdot ds$

$dA = 12(18 \text{ in})(0.03) = \pm 6.48 \text{ in}^2$

$\text{Rel. Err.} = \dfrac{6.48}{6(18)^2} = 0.00\overline{3} \Rightarrow 0.\overline{3}\%$

EX2 → The radius of a sphere is 10 cm w/ a possible error of 0.01 cm. Use differentials to approximate error in volume, surface area, & relative error.

$V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2\, dr$

$dV = 4\pi (10\text{ cm})^2 (0.01\text{ cm})$

$dV = \pm 4\pi\text{ cm}^3$

Rel. Error $= \dfrac{4\pi}{\frac{4}{3}\pi(10)^3} = 0.003 \Rightarrow 0.3\%$

$A = 4\pi r^2$

$dA = 8\pi r\, dr$

$dA = 8\pi (10\text{ cm})(0.01)$

$dA = \pm 0.8\pi\text{ cm}^2$

Rel. Error $= \dfrac{0.8\pi}{4\pi(10)^2} = 0.002 \Rightarrow 0.2\%$

HW: p. 240 → 2-20 even, 27-30, 33, 35, 36