# Extrema on an Interval

- Extreme Value Theorem
  - → If $f$ is continuous on <u>closed</u> interval $[a,b]$, then $f$ has both a minimum AND a maximum on the interval
    - If $f(c) \geq f(x)$ for all $x$ in interval, <u>maximum</u> located @ $(c, f(c))$
    - If $f(c) \leq f(x)$ for all $x$ in interval, <u>minimum</u> located @ $(c, f(c))$
    - Collectively, maxima + minima are called <u>extrema</u>

<u>NOTE:</u> Absolute extrema can only be guaranteed on closed intervals. They are possible on open intervals, but are more likely to be <u>local</u> or <u>relative</u> extrema. Absolute extrema are also considered local or relative extrema.

- To find where an extrema could be found, consider the endpoints (if a closed interval) or critical numbers
  - ↳ Let $f$ be defined at $c$. If $f'(c)=0$ OR does not exist, then $c$ is a critical number of $f$

NOTE: Relative extrema only occur at critical numbers

EX1 → Find the extrema of $y=3x^3-2x^2$ on $[-2,4]$

$y'=9x^2-4x=0$

$x(9x-4)=0$

$x=0, \frac{4}{9}$

$y(-2)=3(-2)^3-2(-2)^2=-24-8=-32$ ← absolute min

$y(0)=0$

$y\left(\frac{4}{9}\right)=3\left(\frac{4}{9}\right)^3-2\left(\frac{4}{9}\right)^2=-\frac{32}{243}$

$y(4)=3(4)^3-2(4)^2=192-32=160$ ← absolute max

EX2 → Find extrema of $y = 6x - 4\sqrt{x}$ on $[0, 9]$

$y' = 6 - 2x^{-1/2} = 0$

$\frac{2}{\sqrt{x}} = 6$

$2 = 6\sqrt{x}$

$\sqrt{x} = \frac{2}{6} = \frac{1}{3}$

$x = \frac{1}{9}$

$y(0) = 0$

$y\left(\frac{1}{9}\right) = 6\left(\frac{1}{9}\right) - 4\left(\sqrt{\frac{1}{9}}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}$ ← absolute min

$y(9) = 6(9) - 4\sqrt{9} = 54 - 12 = 42$ ← absolute max

__HW__: p. 169 → 1-39 odd

(15) $g(t) = t\sqrt{4-t}, \; t > 3$

$g'(t) = t\left(\frac{1}{2}(4-t)^{-1/2} \cdot -1\right) + \sqrt{4-t} = 0$

$\frac{-t}{2\sqrt{4-t}} + \sqrt{4-t} = 0$

$2(\sqrt{4-t})\sqrt{4-t} = \frac{t}{2\sqrt{4-t}} \cdot 2(\sqrt{4-t})$

$2(4-t) = t$

$8 - 2t = t$

$3t = 8$

$t = \frac{8}{3}$

(17) $h(x) = \sin^2 x + \cos x, \; 0 < x < 2\pi$ $\quad \sin^2 x \to (\sin x)^2$

$h'(x) = 2\sin x \cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$

$x = \pi$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

~~11~~

~~15~~

~~17~~

~~25~~

31

33 37

35

39

(25) $y = 3x^{2/3} - 2x$

$y' = 2x^{-1/3} - 2 = 0$

$x^{-1/3} = 1$

$\frac{1}{x^{1/3}} = 1 \quad x = 1$

$y(-1) = 3(-1)^{2/3} - 2(-1) = 5 \leftarrow$ Abs max

$y(1) = 3(1)^{2/3} - 2(1) = 1 \leftarrow$ Abs min

(31) $y(-1) = 3 - |-1-3| = -1 \leftarrow$ abs min

$y(3) = 3 - |3-3| = 3 \leftarrow$ abs max

$y(5) = 3 - |5-3| = 1$

(33) $f(x) = \cos \pi x, \quad [0, \frac{1}{6}]$

$f'(x) = -\pi \underline{\sin \pi x} = 0$

$\sin \pi x = 0 \quad @ x = 0$

$f(0) = \cos 0 = 1 \leftarrow$ abs max

$f(\frac{1}{6}) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \leftarrow$ abs min

(35) $y = \overset{\to 4x^{-1}}{\frac{4}{x}} + \tan\left(\frac{\pi x}{8}\right), \quad [1, 2]$

$y' = \frac{-4}{x^2} + \frac{\pi}{8} \sec^2\left(\frac{\pi x}{8}\right) = 0$

none in domain

$y(1) = \frac{4}{1} + \tan\left(\frac{\pi}{8}\right) = 4 + 0.414 = 4.414 \leftarrow$ abs max

$y(2) = \frac{4}{2} + \tan\left(\frac{\pi}{4}\right) = 2 + 1 = 3 \leftarrow$ abs min

(39) $y = x^2 - 2x$

$y' = 2x - 2 = 0$

$x = 1$

HW: p. 169 → 2-36 even, 55-58

⑥ $f(x) = -3x\sqrt{x+1}$

$$f'(x) = (-3x)\left(\tfrac{1}{2}(x+1)^{-1/2}\right) + \left(\sqrt{x+1}\right)(-3)$$

$$f'(x) = \frac{-3x}{2\sqrt{x+1}} - 3\sqrt{x+1}$$

$$f'\left(-\tfrac{2}{3}\right) = \frac{-3\left(-\frac{2}{3}\right)}{2\sqrt{-\frac{2}{3}+1}} - 3\sqrt{-\tfrac{2}{3}+1} = \frac{\cancel{2}}{\cancel{2}\sqrt{\frac{1}{3}}} - 3\sqrt{\tfrac{1}{3}}$$

$$= \sqrt{3} - 3\cdot\frac{\sqrt{3}}{3} = \sqrt{3} - \sqrt{3} = 0$$

⑯ $f(x) = \dfrac{4x}{x^2+1}$

$$f'(x) = \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2}$$

$$= \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{-4x^2+4}{(x^2+1)^2}$$

$$= \frac{-4(x^2-1)}{(x^2+1)^2} = 0 \Rightarrow -4(x+1)(x-1)$$

$$\boxed{x = \pm 1}$$

⑱ $f(\theta) = 2\sec\theta + \tan\theta,\ 0 < \theta < 2\pi$

$$f'(\theta) = 2\sec\theta\tan\theta + \sec^2\theta = 0$$

$$2\sec\theta\tan\theta = -\sec^2\theta$$

$$2\tan\theta = -\sec\theta$$

$$\cancel{\cos\theta}\cdot 2\frac{\sin\theta}{\cancel{\cos\theta}} = \frac{-1}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}$$

$$2\sin\theta = -1$$

$$\sin\theta = \frac{-1}{2}$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

⑳ $f(x) = \dfrac{2x+5}{3} = \dfrac{2}{3}x + \dfrac{5}{3}$

$$f'(x) = \frac{2}{3}$$

$f(0) = \frac{5}{3}$ ← abs min

$f(5) = \frac{10}{3} + \frac{5}{3} = \frac{15}{3} = 5$ ← abs max

(28) $f(x) = \dfrac{2x}{x^2+1}$, $[-2, 2]$

$$f'(x) = \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2}$$

$$= \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2} = \frac{-2(x^2-1)}{(x^2+1)^2} = 0$$

$$x = \pm 1$$

$f(-2) = \dfrac{-4}{5}$

$f(-1) = \dfrac{-2}{2} = -1$ ← abs min

$f(1) = \dfrac{2}{2} = 1$ ← abs max

$f(2) = \dfrac{4}{5}$

(36) $y = x^2 - 2 - \cos x$, $[-1, 3]$

$y' = 2x + \sin x = 0$

$x = 0$

$f(-1) = 1 - 2 - 0.540 = -1.540$

$f(0) = -2 - 1 = -3$ ← abs min

$f(3) = 9 - 2 + 0.990 = 7.990$ ← abs max

- Rolle's Theorem

→ Let $f$ be continuous on the closed interval $[a,b]$ & differentiable on $(a,b)$. If $f(a)=f(b)$, then there must contain a value in the interval where $f'(x)=0$

EX1 → Using Rolle's Theorem, find the values in the interval $(-2,2)$ when $f'(c)=0$,

$$f(x)=x^4-2x^2$$

$$f(-2)=(-2)^4-2(-2)^2=16-8=8$$

$$f(2)=(2)^4-2(2)^2=16-8=8$$

$$f'(x)=4x^3-4x=0$$

$$4x(x^2-1)=0$$

$$\boxed{x=0,\pm1}$$

EX2 $\Rightarrow$ Determine whether Rolle's Thm can be used & find values where $f'(c)=0$

1) $f(x) = \dfrac{x^2-1}{x}$, $(-1,1)$ $\rightarrow$ NO $\Rightarrow$ not continuous @ $x=0$

2) $f(x) = x^{2/3} - 1$, $(-8,8)$ $\rightarrow$ NO $\Rightarrow$ not differentiable @ $x=0$

3) $f(x) = x^2 - 5x + 4$, $(1,4)$

$f(1) = 1 - 5 + 4 = 0$

$f(4) = 16 - 20 + 4 = 0$

$f'(x) = 2x - 5 = 0$

$x = \frac{5}{2}$

# Mean Value Theorem

→ Let $f(x)$ be continuous on $[a,b]$ & differentiable on $(a,b)$. There exists some $x=c$ in the interval where $f'(c) = \frac{f(b)-f(a)}{b-a}$

EX1 → $f(x) = \frac{4}{x}$, $(-1,3)$ → MVT cannot be used

$f(x) = \frac{4}{x}$, $(1,3)$

$f(1) = \frac{4}{1} = 4$

$f(3) = \frac{4}{3}$

$f'(c) = \frac{f(3)-f(1)}{3-1} = \frac{\frac{4}{3}-4}{2} = -\frac{4}{3}$

$f'(x) = \frac{-4}{x^2} = -\frac{4}{3}$

$x^2 = 3$

$x = \sqrt{3}$

EX2 $\rightarrow f(x)=\sqrt{2-x}\,,\ (-7,2)$

$$f'(c)=\frac{f(2)-f(-7)}{2-(-7)}=\frac{0-3}{9}=-\frac{1}{3}$$

$$f'(x)=\frac{1}{2}(2-x)^{-1/2}\cdot -1=\frac{-1}{2\sqrt{2-x}}=-\frac{1}{3}$$

$$2\sqrt{2-x}=3$$

$$\sqrt{2-x}=\frac{3}{2}$$

$$2-x=\frac{9}{4}$$

$$-x=\frac{1}{4}\Rightarrow x=\frac{-1}{4}$$

2B) $\rightarrow$ What is eqn of tangent line?

$$f\left(-\frac{1}{4}\right)=\sqrt{2-\left(-\frac{1}{4}\right)}=\sqrt{\frac{9}{4}}=\frac{3}{2}$$

$$\boxed{y-\frac{3}{2}=-\frac{1}{3}\left(x+\frac{1}{4}\right)}$$

HW: p. 176 → 1-23 odd, 39-45 odd