Implicit Differentiation

- Use when x can <u>NOT</u> be explicitly solved for (i.e., not convenient to solve for y)

<u>EX1</u> → $\frac{d}{dx}\left[y^3\right] = 3y^2 \cdot \frac{dy}{dx}$ $\left(\frac{d}{dy}\left[y^3\right] \cdot \frac{dy}{dx}\right)$

<u>EX2</u> → $\frac{d}{dx}\left[4x^4 + 5y^2\right] = 16x^3 + 10y \cdot \frac{dy}{dx}$

- To find $\frac{dy}{dx}$ using implicit differentiation
  1) Take derivative w/ respect to x
  2) Factor out $\frac{dy}{dx}$
  3) Divide out $\frac{dy}{dx}$

EX → $y^3 + y^2 - x^2 + 7 = 5$

$\frac{d}{dx}\left[y^3 + y^2 - x^2 + 7\right] = \frac{d}{dx}[5]$

$3y^2 \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} - 2x = 0$

$\frac{dy}{dx}(3y^2 + 2y) = 2x$

$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y}$

EX → $\frac{d}{dx}[4xy] = \frac{d}{dx}[(4x) \cdot y]$

$= 4x \cdot \frac{dy}{dx} + y \cdot 4$

$= 4x \cdot \frac{dy}{dx} + 4y$

EX → Find $\frac{dy}{dx}$ of $x^2 + 4y^2 = 4$ @ $(\sqrt{2}, -\frac{\sqrt{2}}{2})$

$2x + 8y \cdot \frac{dy}{dx} = 0$

$8y \cdot \frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y}$

$\frac{dy}{dx} = \frac{-\sqrt{2}}{4(-\frac{\sqrt{2}}{2})} = \frac{-\sqrt{2}}{-2\sqrt{2}} = \frac{1}{2}$

$\cos(xy) = 1 - x$

$-\sin(xy) \cdot \left[x \cdot \frac{dy}{dx} + y \cdot 1\right] = -1$

$\frac{dy}{dx}(-x\sin(xy)) - y\sin(xy) = -1$

$\frac{dy}{dx}(-x\sin(xy)) = y\sin(xy) - 1$

$\frac{dy}{dx} = \frac{y\sin(xy) - 1}{-x\sin(xy)}$

HW: p. 146 → 1-27 odd, 41
(omit 17, 19)

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$$x^3y^3 - y = x$$

$$x^3 \cdot 3y^2 \cdot \frac{dy}{dx} + y^3 \cdot 3x^2 - \frac{dy}{dx} = 1$$

$$\frac{dy}{dx}\left(3x^3y^2 - 1\right) = 1 - 3x^2y^3$$

$$\frac{dy}{dx} = \frac{1-3x^2y^3}{3x^3y^2-1}$$

15

$$y = \sin(xy)$$

$$\frac{dy}{dx} = \cos(xy) \cdot \left[x \cdot \frac{dy}{dx} + y\right]$$

$$\frac{dy}{dx} = x\cos(xy) \cdot \frac{dy}{dx} + y\cos(xy)$$

$$\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = y\cos(xy)$$

$$\frac{dy}{dx}\left(1 - x\cos(xy)\right) = y\cos(xy)$$

$$\frac{dy}{dx} = \frac{y\cos(xy)}{1-x\cos(xy)}$$

25

$$x^{2/3} + y^{2/3} = 5$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot \frac{dy}{dx} = 0$$

$$\frac{2}{3}y^{-1/3} \cdot \frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$$

$$\frac{dy}{dx} = \frac{-\cancel{\frac{2}{3}}x^{-1/3}}{\cancel{\frac{2}{3}}y^{-1/3}} = \frac{-y^{1/3}}{x^{1/3}}$$

27

$$\tan(x+y) = x$$

$$\sec^2(x+y)\left(1 + \frac{dy}{dx}\right) = 1$$

$$\sec^2(x+y) + \sec^2(x+y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1-\sec^2(x+y)}{\sec^2(x+y)} = \frac{1-\sec^2(0)}{\sec^2(0)} = 0$$

41) A) $\frac{1}{2}x^2 + \frac{1}{8}y^2 = 1$

$$x + \frac{1}{4}y \cdot \frac{dy}{dx} = 0$$

$$\frac{1}{4}y \cdot \frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-4x}{y} = \frac{-4}{2} = -2$$

$$\boxed{y-2 = -2(x-1)}$$

B) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2}$$

$$\frac{dy}{dx} = \frac{-2xb^2}{2ya^2} = \frac{-b^2x}{a^2y} = \frac{-b^2x_0}{a^2y_0}$$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

$$y - y_0 = \frac{-b^2x_0}{a^2y_0}(x - x_0)$$

$$a^2y_0y - a^2(y_0)^2 = -b^2x_0x + b^2(x_0)^2$$

$$\frac{a^2y_0y + b^2x_0x}{a^2b^2} = \frac{a^2(y_0)^2 + b^2(x_0)^2}{a^2b^2}$$

$$\frac{y_0y}{b^2} + \frac{x_0x}{a^2} = \frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2}$$

$$\boxed{\frac{y_0y}{b^2} + \frac{x_0x}{a^2} = 1}$$

- Finding 2nd Derivative

EX → $x^2 + y^2 = 4$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y \cdot (-1) - (-x)\left(\frac{dy}{dx}\right)}{y^2}$$

$$= \frac{y(-1) - (-x)\left(\frac{-x}{y}\right)}{y^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-y - \frac{x^2}{y}}{y^2}}$$

HW: p. 146 → 2-28 even (omit 18, 20)
45-50

8

$$\sqrt{xy} = x - 2y$$

$$(xy)^{1/2} = x - 2y$$

$$\frac{1}{2}(xy)^{-1/2} \cdot \left[x \cdot \frac{dy}{dx} + y\right] = 1 - 2\frac{dy}{dx}$$

$$\frac{1}{2}x(xy)^{-1/2} \cdot \frac{dy}{dx} + \frac{1}{2}y(xy)^{-1/2} = 1 - 2\frac{dy}{dx}$$

$$\frac{1}{2}x(xy)^{-1/2}\frac{dy}{dx} + 2\frac{dy}{dx} = 1 - \frac{1}{2}y(xy)^{-1/2}$$

$$\frac{dy}{dx}\left(\frac{1}{2}x(xy)^{-1/2} + 2\right) = 1 - \frac{1}{2}y(xy)^{-1/2}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - \frac{1}{2}y(xy)^{-1/2}}{\frac{1}{2}x(xy)^{-1/2} + 2}}$$

10

$$2\sin x \cos y = 1$$

$$2\sin x \cdot -\sin y \cdot \frac{dy}{dx} + \cos y \cdot 2\cos x = 0$$

$$2\sin x \sin y \cdot \frac{dy}{dx} = 2\cos x \cos y$$

$$\frac{dy}{dx} = \frac{2\cos x \cos y}{2\sin x \sin y} = \frac{\cos x \cos y}{\sin x \sin y} = \cot x \cot y$$

8
~~10~~
12
16
24
40
48

12

$$(\sin \pi x + \cos \pi y)^2 = 2$$

$$2(\sin \pi x + \cos \pi y)\left(\pi \cos \pi x - \pi \sin \pi y \cdot \frac{dy}{dx}\right) = 0$$

$$2\Big(\pi \sin \pi x \cos \pi x - \pi \sin \pi x \sin \pi y \frac{dy}{dx} + \pi \cos \pi x \cos \pi y - \pi \sin \pi y \cos \pi y \frac{dy}{dx}\Big) = 0$$

$$-2\pi \sin \pi x \sin \pi y \frac{dy}{dx} - 2\pi \sin \pi y \cos \pi y \frac{dy}{dx} = -2\pi \sin \pi x \cos \pi x - 2\pi \cos \pi x \cos \pi y$$

$$\boxed{\frac{dy}{dx} = \frac{\sin \pi x \cos \pi x + \cos \pi x \cos \pi y}{\sin \pi x \sin \pi y + \sin \pi y \cos \pi y}}$$

(16) $x = \sec\left(\frac{1}{y}\right)$

$$1 = \sec\left(\tfrac{1}{y}\right)\tan\left(\tfrac{1}{y}\right) \cdot \frac{-1}{y^2} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec\left(\frac{1}{y}\right)\tan\left(\frac{1}{y}\right)\cdot\frac{-1}{y^2}}$$

(24) $(x+y)^3 = x^3 + y^3, \ (-1,1)$

$$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3(x+y)^2 + 3(x+y)^2\frac{dy}{dx} = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3(x+y)^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 3x^2 - 3(x+y)^2$$

$$\frac{dy}{dx}\left((x+y)^2 - y^2\right) = x^2 - (x+y)^2$$

$$\frac{dy}{dx} = \frac{x^2-(x+y)^2}{(x+y)^2-y^2} = \frac{1-0}{0-1} = -1$$

(28) $x\cos y = 1, \ \left(2, \frac{\pi}{3}\right)$

$$x \cdot -\sin y\,\frac{dy}{dx} + \cos y = 0$$

$$\cos y = x\sin y\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\cos y}{x\sin y} = \frac{\cos\left(\frac{\pi}{3}\right)}{2\cdot\sin\left(\frac{\pi}{3}\right)} = \frac{\frac{1}{2}}{\cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

(46) $x^2y^2 - 2x = 3$

$$2xy^2 + 2x^2y\frac{dy}{dx} - 2 = 0$$

$$x^2y\frac{dy}{dx} = 1 - xy^2$$

$$\frac{dy}{dx} = \frac{1-xy^2}{x^2y}$$

$$\frac{d^2y}{dx^2} = \frac{(x^2y)\left(-x\cdot 2y\frac{dy}{dx} - y^2\right) - (1-xy^2)\left(x^2\frac{dy}{dx} + 2xy\right)}{(x^2y)^2}$$

$$= \frac{(x^2y)\left(-2\cancel{xy}\left(\frac{1-xy^2}{x^{\cancel{2}}\cancel{y}}\right) - y^2\right) - (1-xy^2)\left(\cancel{x^2}\cdot\frac{1-xy^2}{\cancel{x^2}y} + 2xy\right)}{x^4y^2}$$

$$= \frac{(x^2y)\left(\frac{-2(1-xy^2)}{x} - y^2\right) - (1-xy^2)\left(\frac{1-xy^2}{y} + 2xy\right)}{x^4y^2}$$

48 $1 - xy = x - y$

$$-x \cdot \frac{dy}{dx} + -y = 1 - \frac{dy}{dx}$$

$$-x \frac{dx}{dx} + \frac{dy}{dx} = 1 + y$$

$$\frac{dy}{dx} = \frac{1+y}{1-x}$$

$$\frac{d^2y}{dx^2} = \frac{(1-x)\left(\frac{dy}{dx}\right) - (1+y)(-1)}{(1-x)^2}$$

$$= \frac{\cancel{(1-x)}\left(\frac{1+y}{\cancel{1-x}}\right) + 1+y}{(1-x)^2} = \frac{2+2y}{(1-x)^2}$$

<u>HW</u>: p. 146 → 29-32, 53, 54, 57, 58, 76-78