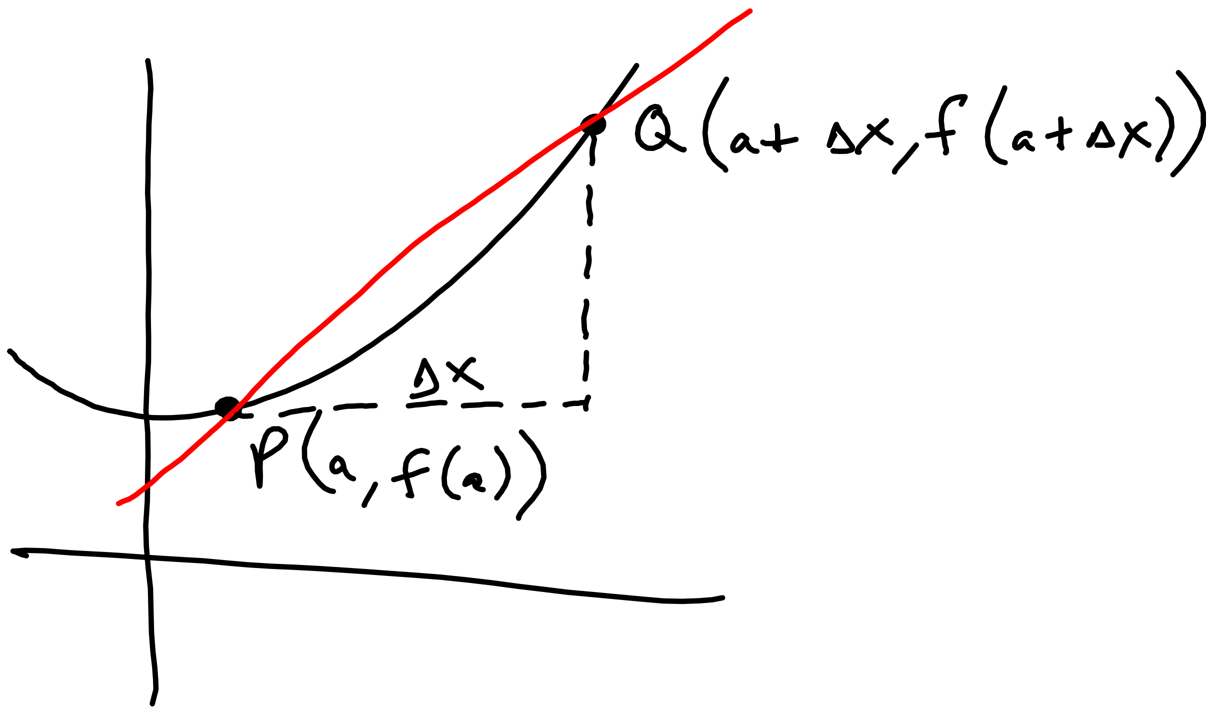
Differentiation

- Let's consider a general curve. Let's then attempt to find the slope of a tangent line at some point on the curve



→ We can approximate the rate of change by the slope of secant line through the desired point + some other point

$$m = \frac{f(a+\Delta x) - f(a)}{(a+\Delta x) - a} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

→ If f is defined on an open interval containing a, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = m$$

exists, then the line passing through $(a, f(a))$ w/ slope m is the tangent line to $f(x)$ at a.

EX1 → Slope of $f(x) = 3x+4$ at $(3,13)$

$$\lim_{\Delta x \to 0} \frac{f(3+\Delta x) - f(3)}{\Delta x} = \frac{3(3+\Delta x)+4-13}{\Delta x}$$

$$= \frac{9+3(\Delta x)+4-13}{\Delta x}$$

$$= \frac{3(\Delta x)+13-13}{\Delta x}$$

$$= \frac{3(\Delta x)}{\Delta x}$$

$$= 3$$

EX2 → Slope of $f(x) = x^2 + 4x - 5$ @ $(2,7)$

$$\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(2+\Delta x)^2 + 4(2+\Delta x) - 5 - 7}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 4(\Delta x) + 4 + 8 + 4(\Delta x) - 12}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 8(\Delta x) + 12 - 12}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 8(\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x + 8 = 0 + 8 = 8$$

EX3 → Slope of $f(x) = x^2 + 4x - 5$ @ $(a, f(a))$

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(a+\Delta x)^2 + 4(a+\Delta x) - 5 - (a^2 + 4a - 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a^2 + 2a(\Delta x) + (\Delta x)^2 + 4a + 4(\Delta x) - 5 - a^2 - 4a + 5}{\Delta x} = \lim_{\Delta x \to 0} \frac{2a(\Delta x) + (\Delta x)^2 + 4(\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2a + \Delta x + 4 = 2a + 4$$

- Let's revisit our first example regarding speed of an object. Notice how the process to find instantaneous speed is the exact same as finding slope of a tangent line

(*) → The limit used to find the slope of a tangent line is also used to define the derivative at $f(a)$

(*) → The derivative of the function $f(x)$ is the function $f'(x)$ whose value at x is

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists

→ If $f'(x)$ exists, then $f(x)$ is differentiable at x

HW: p. 103 → 1, 2, 5-10, 26-36 even

(30) $f(x) = \sqrt{x-1}\ ,\ (5,2)$

$$\lim_{\Delta x \to 0} \frac{f(5+\Delta x) - f(5)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{5+\Delta x-1} - 2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{4+\Delta x} - 2}{\Delta x} \cdot \frac{\sqrt{4+\Delta x} + 2}{\sqrt{4+\Delta x} + 2}$$

$$= \lim_{\Delta x \to 0} \frac{4+\Delta x - 4}{\Delta x\left(\sqrt{4+\Delta x}+2\right)} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{4+\Delta x}+2\right)} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{4+\Delta x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

$$y - y_1 = m(x - x_1) \Rightarrow \boxed{y - 2 = \frac{1}{4}(x-5)}$$

(32) $f(x) = \dfrac{1}{x+1}\ ,\ (0,1)$

$$\lim_{\Delta x \to 0} \frac{f(\cancel{0+\Delta x}\ \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\Delta x+1} - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\Delta x+1} - \frac{(\Delta x+1)}{(\Delta x+1)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1-\Delta x-1}{\Delta x+1}}{\frac{\Delta x}{1}}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(\Delta x+1)} = \lim_{\Delta x \to 0} \frac{-1}{\Delta x+1} = -1$$

$$\boxed{y - 1 = -1(x - 0)}$$

(34) $3x - y - 4 = 0 \Rightarrow y = 3x - 4$

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(a+\Delta x)^3 + 2 - (a^3+2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(a+\Delta x)(a+\Delta x)(a+\Delta x) + 2 - a^3 - 2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(a^2 + 2a(\Delta x) + (\Delta x)^2)(a+\Delta x) - a^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{a^3} + 2a^2(\Delta x) + a(\Delta x)^2 + a^2(\Delta x) + 2a(\Delta x)^2 + (\Delta x)^3 - \cancel{a^3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3a^2(\Delta x) + 3a(\Delta x)^2 + (\Delta x)^3}{\Delta x} = \lim_{\Delta x \to 0} 3a^2 + 3a(\Delta x) + (\Delta x)^2 = 3a^2$$

$3a^2 = 3$

$a^2 = 1$

$a = \pm 1 \Rightarrow (1, 3), (-1, 1)$

$\boxed{y - 3 = 3(x-1)}$ + $\boxed{y - 1 = 3(x+1)}$

(36) $f(x) = \frac{1}{\sqrt{x-1}}$, $x + 2y + 7 = 0 \Rightarrow 2y = -x - 7 \Rightarrow y = -\frac{1}{2}x - \frac{7}{2}$

$$\lim_{\Delta x \to 0} \frac{\frac{1}{\sqrt{a+\Delta x-1}} - \frac{1}{\sqrt{a-1}}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\sqrt{a-1} - \sqrt{a+\Delta x-1}}{\sqrt{a+\Delta x-1}\cdot\sqrt{a-1}}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{a-1} - \sqrt{a+\Delta x-1}}{\Delta x\sqrt{a+\Delta x-1}\cdot\sqrt{a-1}} \cdot \frac{\sqrt{a-1} + \sqrt{a+\Delta x-1}}{\sqrt{a-1} + \sqrt{a+\Delta x-1}}$$

$$= \lim_{\Delta x \to 0} \frac{a-1-(a+\Delta x-1)}{\Delta x\sqrt{a+\Delta x-1}\sqrt{a-1}\left(\sqrt{a-1}+\sqrt{a+\Delta x-1}\right)} = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x\,(\text{STUFF})} = \lim_{\Delta x \to 0} \frac{-1}{\text{STUFF}} = \frac{-1}{\sqrt{a-1}\sqrt{a-1}\left(\sqrt{a-1}+\sqrt{a-1}\right)}$$

$$= \frac{-1}{(a-1)(2\sqrt{a-1})} = \frac{-1}{2(a-1)^{3/2}}$$

$$\frac{-1}{2(a-1)^{3/2}} = \frac{-1}{2} \Rightarrow 2(a-1)^{3/2} = 2$$

$$\left[(a-1)^{3/2} = 1\right]^2$$

$$(a-1)^3 = 1$$

$$a-1 = 1$$

$$a = 2$$

$f(2) = 1 \Rightarrow \boxed{y - 1 = -\frac{1}{2}(x-2)}$

EX1 → Find the derivative of $f(x) = x^2 + 7x + 6$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 7(x+\Delta x) + 6 - (x^2+7x+6)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x(\Delta x) + (\Delta x)^2 + \cancel{7x} + 7(\Delta x) + \cancel{6} - \cancel{x^2} - \cancel{7x} - \cancel{6}}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x(\Delta x) + 7(\Delta x) + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + 7 + \Delta x = \boxed{2x+7}$$

EX2 → Find slope of $f(x) = \sqrt{x+1}$ @ $(3,2)$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+1} - \sqrt{x+1}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+1} + \sqrt{x+1}}{\sqrt{x+\Delta x+1} + \sqrt{x+1}} = \lim_{\Delta x \to 0} \frac{x+\Delta x+1 - (x+1)}{\Delta x\left(\sqrt{x+\Delta x+1} + \sqrt{x+1}\right)} =$$

$$= \lim_{\Delta x \to 0} \frac{x+\Delta x+1-x-1}{\Delta x\left(\sqrt{x+\Delta x+1} + \sqrt{x+1}\right)} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x+1} + \sqrt{x+1}\right)} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+1} + \sqrt{x+1}} = \frac{1}{\sqrt{x+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

$$f'(3) = \frac{1}{2\sqrt{3+1}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

Eqn. of Line $\Rightarrow \boxed{y - 2 = \frac{1}{4}(x-3)}$

- Notation
  - $\to f'(x) \to$ "f prime of x"
  - $\to y' \to$ "y prime"
  - $\to \frac{dy}{dx} \to$ "dy dx" (derivative of y w/ respect to x)
  - $\to \frac{df}{dx} \to$ "df dx" (derivative of f w/ respect to x)
  - $\to \frac{d}{dx}[f(x)] \to$ "d dx of f of x" (derivative of f of x (emphasizes derivative of function))

- Alternate Definition of Derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{(Derivative at point)}$$

EX → Find the derivative of $f(x) = \sqrt{x}$ at $x = a$ using the alternative def'n.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}$$

$$= \lim_{x \to a} \frac{\cancel{x - a}}{\cancel{x - a}(\sqrt{x} + \sqrt{a})} = \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

HW: p. 104 → 11–35 odd, 73–78 (omit 75)