Logarithmic Functions

- Derivative of Logarithmic Functions

  (a is a positive real number $\neq 1$, u is a differentiable function of x)

  $\rightarrow \frac{d}{dx}\left[a^u\right] = \ln a \cdot a^u \cdot \frac{du}{dx}$

EX1 $\rightarrow \frac{d}{dx}\left[4^{5x}\right] = \ln 4 \cdot 4^{5x} \cdot 5 = 5\ln 4 \cdot 4^{5x}$

EX2 $\rightarrow \frac{d}{dx}\left[10^x\right] = \ln 10 \cdot 10^x \cdot 1 = \ln 10 \cdot 10^x$

$\rightarrow \frac{d}{dx}\left[\log_a u\right] = \frac{1}{\ln a \cdot u} \cdot \frac{du}{dx}$

$\log_a u = \frac{\ln u}{\ln a} \Rightarrow \frac{d}{dx}\left[\frac{\ln u}{\ln a}\right] = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}$

<u>EX3</u> $\rightarrow \frac{d}{dx}\left[\log_7 10x^2\right] = \frac{1}{\ln 7} \cdot \frac{1}{10x^2} \cdot 20x = \frac{2}{\ln 7 \cdot x}$

<u>EX4</u> $\rightarrow \frac{d}{dx}\left[\log_5 8x\right] = \frac{1}{\ln 5} \cdot \frac{1}{8x} \cdot 8 = \frac{1}{\ln 5 \cdot x}$

<u>HW</u>: p. 366 $\rightarrow$ 3-57 mult. 3 (omit 9, 12, 33, 36)

39) $g(t) = t^2 \cdot 2^t$

$$g'(t) = t^2 \cdot \ln 2 \cdot 2^t \cdot 1 + 2^t \cdot 2t$$

$$\boxed{g'(t) = \ln 2 \cdot t^2 \cdot 2^t + 2t \cdot 2^t}$$

48) $f(t) = t^{3/2} \cdot \log_2 \sqrt{t+1} \rightarrow (t+1)^{1/2}$

$$f'(t) = t^{3/2} \cdot \frac{1}{\ln 2 \cdot \sqrt{t+1}} \cdot \frac{1}{2}(t+1)^{-1/2} + \log_2 \sqrt{t+1} \cdot \frac{3}{2}t^{1/2}$$

$$= \frac{t^{3/2}}{2\ln 2 \sqrt{t+1}} \cdot \frac{1}{\sqrt{t+1}} + \frac{3}{2}t^{1/2} \cdot \log_2 \sqrt{t+1}$$

$$\boxed{f'(t) = \frac{t^{3/2}}{2\ln 2 (t+1)} + \frac{3}{2}t^{1/2} \cdot \log_2 \sqrt{t+1}}$$

42) $g(\alpha) = 5^{-\frac{1}{2}\alpha} \cdot \sin 2\alpha$

$$g'(\alpha) = 5^{-\frac{1}{2}\alpha} \cdot \cos 2\alpha \cdot 2 + \sin 2\alpha \cdot \ln 5 \cdot 5^{-\frac{1}{2}\alpha} \cdot -\frac{1}{2}$$

$$\boxed{g'(\alpha) = 5^{-\frac{1}{2}\alpha} \cdot 2\cos 2\alpha - \frac{1}{2}\ln 5 \cdot \sin 2\alpha \cdot 5^{-1/2\alpha}}$$

54) $y = x^{x-1} \Rightarrow \ln y = \ln x^{x-1}$

$$\ln y = (x-1) \cdot \ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = (x-1) \cdot \frac{1}{x} + \ln x \cdot 1$$

$$\frac{dy}{dx} = y\left(1 - \frac{1}{x} + \ln x\right)$$

$$\boxed{\frac{dy}{dx} = \left(x^{x-1}\right)\left(1 - \frac{1}{x} + \ln x\right)}$$

57) $y = x^{\sin x}, \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\ln y = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \cdot \frac{1}{x} + \ln x \cdot \cos x$$

$$\frac{dy}{dx} = y\left(\frac{\sin x}{x} + \ln x \cdot \cos x\right)$$

$$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \ln x \cdot \cos x\right)$$

$$\frac{dy}{dx} = \left(\frac{\pi}{2}\right)^{\sin\left(\frac{\pi}{2}\right)}\left(\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} + \ln\left(\frac{\pi}{2}\right) \cdot \cos\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{\pi}{2}^{1}\left(\frac{1}{\frac{\pi}{2}} + 0\right) = 1$$

$$\boxed{y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)}$$

- Integrating Logarithmic Functions

$$\rightarrow \int a^x \, dx = \left(\frac{1}{\ln a}\right) a^x + C$$

EX1 $\rightarrow \int (3-x) \cdot 7^{(3-x)^2} dx$

$u = (3-x)^2$

$du = 2(3-x) \cdot -1 \, dx$

$-\frac{1}{2} du = (3-x) dx$

$$= \int 7^u \cdot -\frac{1}{2} du = \int -\frac{1}{2} \cdot 7^u du = -\frac{1}{2} \cdot \frac{1}{\ln 7} \cdot 7^u + C = \boxed{-\frac{1}{2\ln 7} \cdot 7^{(3-x)^2} + C}$$

EX2 $\rightarrow \int \cos x \cdot 2^{\sin x} dx$

$u = \sin x$

$du = \cos x \, dx$

$$= \int 2^u du$$

$$= \frac{1}{\ln 2} \cdot 2^u + C = \boxed{\frac{1}{\ln 2} \cdot 2^{\sin x} + C}$$

HW: p. 366 → 40-60 mult. 4, 61-72

(44) $f(x) = \log_3 \frac{x\sqrt{x-1}}{2}$

$= \log_3 x + \log_3 \sqrt{x-1} - \log_3 2$

$= \log_3 x + \frac{1}{2}\log_3 (x-1) - \log_3 2$

$\vdots$

(60) $y = x^{1/x}, \ (1,1)$

$\ln y = \ln x^{1/x} = \frac{1}{x} \cdot \ln x$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{x} + \ln x \cdot \frac{-1}{x^2}$

$\frac{dy}{dx} = y\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)$

$\frac{dy}{dx} = 1\left(\frac{1}{1} - \frac{\cancel{\ln 1}^{\,0}}{1}\right)$

$\frac{dy}{dx} = 1$

$y - 1 = 1(x-1)$

(56) $y = (1+x)^{1/x}$

$\ln y = \ln (1+x)^{1/x}$

$\ln y = \frac{1}{x} \cdot \ln(1+x)$

$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{1+x} + \ln(1+x) \cdot \frac{-1}{x^2}$

$\frac{dy}{dx} = y\left(\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right)$

$\frac{dy}{dx} = (1+x)^{1/x}\left(\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right)$

60

65

67

(65) $\displaystyle\int \frac{3^{2x}}{1+3^{2x}}\,dx$

$u = 1+3^{2x}$

$du = \ln 3 \cdot 3^{2x} \cdot 2\,dx$

$du = \frac{2}{\ln 3} \cdot 3^{2x}\,dx$

$\frac{\ln 3}{2}\,du = 3^{2x}\,dx$

$= \displaystyle\int \frac{1}{u} \cdot \frac{\ln 3}{2}\,du$

$= \frac{\ln 3}{2} \cdot \ln|u| + C = \boxed{\frac{\ln 3}{2} \cdot \ln\left|1+3^{2x}\right| + C}$

(67) $\displaystyle\int_{-1}^{2} 2^x\,dx$

$= \frac{1}{\ln 2} \cdot 2^x \Big|_{-1}^{2}$

$= \frac{1}{\ln 2} \cdot 2^2 - \frac{1}{\ln 2} \cdot 2^{-1}$

$= \frac{4}{\ln 2} - \frac{1}{2\ln 2} = \boxed{\frac{7}{2\ln 2}}$

- Applications of Logarithmic Functions → Compound Interest

P = Principal (amount of deposit)

t = number of yrs.

A = balance after t yrs.

r = interest rate

n = # of compoundings

→ Yearly ⇒ $A = Pe^{rt}$

→ Continuously ⇒ $A = P\left(1+\frac{r}{n}\right)^{nt}$

<u>EX 1</u> → $P = \$10,000$, $t = 5$ yrs, $r = 5\%$ (yearly)

$$A = 10000 \cdot e^{(0.05)(5)}$$

$$\boxed{A = \$12,840.30}$$

<u>EX 2</u> → $P = \$10,000$, $t = 5$ yrs, $n = 12$

$r = 5\%$ (compounded monthly for 5 yrs)

$$A = 10000\left(1 + \frac{0.05}{12}\right)^{(12)(5)}$$

$$\boxed{A = \$12,833.60}$$

HW: p. 367 → 79–93 odd