Transcendental Functions

Natural Logs

- Definition: $\ln(x) = \int_1^x \frac{1}{t}\,dt$, $x > 0$

- Properties
  - → Domain: $(0, \infty)$
  - → Range: $(-\infty, \infty)$
  - → $\frac{d}{dx}[\ln x] = \frac{1}{x}$
    - ↳ Since $\frac{d}{dx}$ is always positive, $\ln x$ is <u>always</u> increasing
      - ↳ 1-to-1 function (each x has 1 y)
    - ↳ Implies continuity
  - → $\frac{d^2}{dx^2}[\ln x] = \frac{-1}{x^2}$
    - ↳ Since $\frac{d^2}{dx^2}$ is always negative, $\ln x$ is concave down

- Logarithm Properties (also applies to "regular" logarithms)

1) $\ln(1) = 0$
2) $\ln(a \cdot b) = \ln a + \ln b$
3) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
4) $\ln(a^b) = b \cdot \ln a$

<u>EX1</u> → $\ln(72) = \ln(8) + \ln(9) = \ln(2^3) + \ln(3^2) = 3\ln 2 + 2\ln 3$

<u>EX2</u> → $\ln\left(\frac{4}{5}\right) = \ln 4 - \ln 5$

<u>EX3</u> → $\ln\left(\frac{x^2}{\sqrt{3x+7}}\right) = \ln(x^2) - \ln\left(\sqrt{3x+7}\right) = 2\ln(x) - \frac{1}{2}\ln(3x+7)$

<u>HW</u>: p. 329 → 7-10, 17-33 odd, 37-55 odd

- Natural number e

↳ Definition: $\ln e = \int_1^e \frac{1}{t}\,dt = 1 \quad (e \approx 2.718)$

- Derivative of Natural Log Function

→ $\frac{d}{dx}[\ln(u)] = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

EX 1 → $\frac{d}{dx}[\ln(4x)] = \frac{1}{4x} \cdot 4 = \frac{4}{4x} = \boxed{\frac{1}{x}}$

EX 2 → $\frac{d}{dx}[x^2 \cdot \ln(x)] = x^2 \cdot \frac{1}{x} + \ln(x) \cdot 2x = \boxed{x + 2x\ln(x)}$

EX 3 → $\frac{d}{dx}\left[\ln\left(\frac{x^2(x^3+4)^4}{\sqrt{2x^3-1}}\right)\right] = \frac{d}{dx}\left[\ln(x^2) + \ln(x^3+4)^4 - \ln\left(\sqrt{2x^3-1}\right)\right]$

$= \frac{d}{dx}\left[2\ln x + 4\ln(x^3+4) - \frac{1}{2}\ln(2x^3-1)\right] = 2 \cdot \frac{1}{x} + 4 \cdot \frac{1}{x^3+4} \cdot 3x^2 - \frac{1}{2} \cdot \frac{1}{2x^3-1} \cdot 6x^2$

$= \boxed{\frac{2}{x} + \frac{12x^2}{x^3+4} - \frac{3x^2}{2x^3-1}}$

- Another technique that can be used to differentiate particularly "messy" functions is logarithmic differentiation. To do so, you must "log" both sides

EX » $y = \dfrac{(x-2)^2}{\sqrt{x^2-1}}$, $x \neq 2$

$$\ln y = \ln\left(\frac{(x-2)^2}{\sqrt{x^2-1}}\right) = \ln(x-2)^2 - \ln\left(\sqrt{x^2-1}\right) = 2\ln(x-2) - \tfrac{1}{2}\ln(x^2-1)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x-2} \cdot 1 - \frac{1}{2} \cdot \frac{1}{x^2-1} \cdot 2x$$

$$\cancel{y} \cdot \frac{1}{\cancel{y}} \cdot \frac{dy}{dx} = \frac{2}{x-2} - \frac{x}{x^2-1} \cdot y$$

$$\frac{dy}{dx} = y\left(\frac{2}{x-2} - \frac{x}{x^2-1}\right) = \boxed{\frac{(x-2)^2}{\sqrt{x^2-1}}\left(\frac{2}{x-2} - \frac{x}{x^2-1}\right)}$$

HW: p. 330 → 48, 56, 68, 71, 75, 79, 93-97 odd

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$$y = \ln\left(\sqrt{2+\cos^2 x}\right)$$

$$y = \frac{1}{2}\ln\left(2+\cos^2 x\right) \quad \cos^2 x \to (\cos x)^2$$

$$y' = \frac{1}{\cancel{2}}\cdot\frac{1}{2+\cos^2 x}\cdot -\cancel{2}\cos x\sin x$$

$$\boxed{y' = \frac{-\cos x \sin x}{2+\cos^2 x}}$$

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$$f(x) = 3x^2 - \ln x, \quad (1,3)$$

$$f'(x) = 6x - \frac{1}{x}$$

$$f'(1) = 6 - 1 = \underline{5}$$

$$\boxed{y - 3 = 5(x-1)}$$

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$$y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}} \Rightarrow \ln y = \ln\left(\frac{x(x-1)^{3/2}}{\sqrt{x+1}}\right) = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{3}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1} \Rightarrow \frac{dy}{dx} = y\left(\frac{1}{x} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}\left(\frac{1}{x} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right)}$$

79) $x+y-1=\ln(x^2+y^2)$, $(1,0)$

$$1+\frac{dy}{dx}=\frac{1}{x^2+y^2}\cdot 2x+2y\frac{dy}{dx}$$

$$1+\frac{dy}{dx}=\frac{2x+2y\frac{dy}{dx}}{x^2+y^2}$$

$$1+\frac{dy}{dx}=\frac{2(1)\cancel{+2(0)\frac{dy}{dx}}\to 0}{(1)^2+(0)^2}$$

$$1+\frac{dy}{dx}=\frac{2}{1}=2$$

$$\frac{dy}{dx}=1$$

$$\boxed{y-0=1(x-1)}$$

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- Integrating Natural Log Functions

→ $\int \frac{1}{x}\,dx = \ln|x| + C$

→ $\int \frac{u'}{u}\,du = \ln|u| + C$

EX1 → $\int \frac{1}{5x+2}\,dx$

$u = 5x+2$

$du = 5dx \Rightarrow \frac{1}{5}du = dx$

$= \int \frac{1}{5} \cdot \frac{1}{u}\,du = \frac{1}{5}\ln|u| + C = \boxed{\frac{1}{5}\ln|5x+2| + C}$

EX2 → $\int \frac{5}{9x-4}\,dx$

$u = 9x-4$

$du = 9\,dx \Rightarrow \frac{5}{9}du = 5dx$

$= \int \frac{5}{9} \cdot \frac{1}{u}\,du = \frac{5}{9}\ln|u| + C = \boxed{\frac{5}{9}\ln|9x-4| + C}$

EX3 → $\int_0^3 \frac{x}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \int_{x=0}^{x=3} \frac{1}{2} \cdot \frac{1}{u}\,du$

$= \int_{u=1}^{u=10} \frac{1}{2} \cdot \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| \Big|_{u=1}^{u=10}$

$= \frac{1}{2}\ln 10 - \cancel{\frac{1}{2}\ln 1} \xrightarrow{} 0 = \boxed{\frac{1}{2}\ln 10}$

- Using Natural Logs to Integrate Trig Functions

$\rightarrow \int \tan x \, dx = -\ln|\cos x| + C$

$= \int \frac{\sin x}{\cos x} dx$ $\quad u = \cos x$, $du = -\sin x \, dx$

$= \int -\frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$

$\rightarrow \int \cot x \, dx = \ln|\sin x| + C$

HW: p. 338 → 1-39 odd, 47-53 odd

$\rightarrow \int \sec x \, dx = \ln|\sec x + \tan x| + C$

$= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

$u = \sec x + \tan x$
$du = \tan x \sec x + \sec^2 x \, dx$

$= \int \frac{1}{u} du = \ln|u| + C = \ln|\sec x + \tan x| + C$

$\rightarrow \int \csc x \, dx = -\ln|\csc x + \cot x| + C$