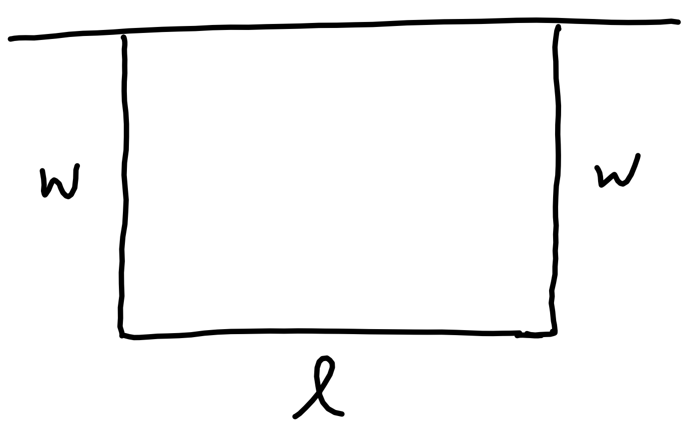
Optimization

→ applied extrema problems

- Steps to solve
  1) Identify all given & wanted quantities (sketches also help!)
  2) Determine equation for quantity to be determined & express it in terms of one variable (similar to related rates, other quantities may be needed)
     ↳ Determine a reasonable domain, creating a closed interval
  3) Solve using analytical techniques (1st/2nd derivative tests, etc.)

EX 1 → A farmer's fence is enclosed on one side by a barn, and is a total of 250 ft. in length. What is the maximum area enclosed by the fence?



$P = \ell + 2w = 250 \Rightarrow \ell = 250 - 2w$, w is $[0, 125]$

$A = \ell \cdot w$

$A = (250 - 2w) \cdot w = 250w - 2w^2$

$A' = 250 - 4w = 0$

$4w = 250$

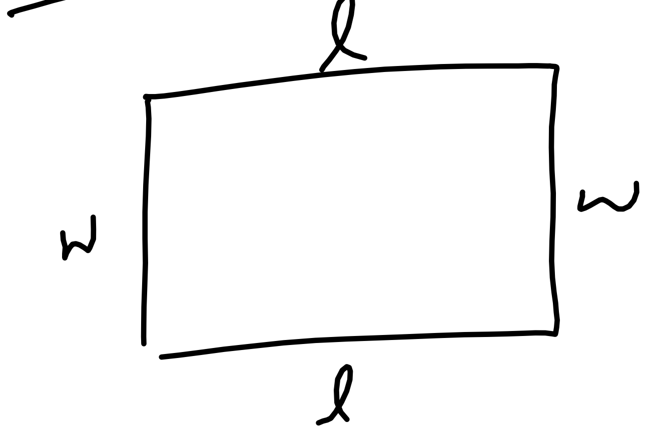
$w = \frac{250}{4} = \frac{125}{2} = 62.5$

$A(0) = (250)(0) = 0$

$A(62.5) = (125)(62.5) = \boxed{78125 \text{ ft}^2}$

$A(125) = (0)(125) = 0$

EX2 → What is maximum area of a rectangle w/ a perimeter of 144 ft ?



$P = 2\ell + 2w = 144 \Rightarrow 2w = 144 - 2\ell \Rightarrow w = 72 - \ell$, $\ell\,[0, 72]$

$A = \ell \cdot w = \ell(72 - \ell) = 72\ell - \ell^2$

$A' = 72 - 2\ell = 0$

$2\ell = 72$

$\ell = 36$

$A(0) = (0)(72) = 0$

$A(36) = (36)(36) = \boxed{1296 \text{ ft}^2}$

$A(72) = (72)(0) = 0$

EX3 → If the product of 2 numbers is 192, what is the smallest possible sum of the 2 numbers?

$P = x \cdot y = 192 \Rightarrow x = \dfrac{192}{y}$, $y\,[1, 192]$

$S = x + y = \dfrac{192}{y} + y$

$S' = -\dfrac{192}{y^2} + 1 = 0$

$1 = \dfrac{192}{y^2}$

$y^2 = 192$

$y = \sqrt{192}$ (assume positive)

$y = 8\sqrt{3}$

$S(1) = \dfrac{192}{1} + 1 = 192 + 1 = 193$

$S(8\sqrt{3}) = \dfrac{192}{8\sqrt{3}} + 8\sqrt{3} = 8\sqrt{3} + 8\sqrt{3} = \boxed{16\sqrt{3}}$

$S(192) = \dfrac{192}{192} + 192 = 1 + 192 = 193$

HW: p. 223 → 3-12, 18-22

(6) $S = x + \frac{1}{x}$

$S' = 1 - \frac{1}{x^2} = 0$

$\frac{1}{x^2} = 1$

$x^2 = 1$

$x = \pm 1 \Rightarrow x = 1$

$\frac{1}{1} = 1$

(8) $x^2 + y = 27 \Rightarrow y = 27 - x^2$, x $[0, 3\sqrt{3}]$

$P = x \cdot y = x(27 - x^2) = 27x - x^3$

$P' = 27 - 3x^2 = 0$

$3x^2 = 27$

$x^2 = 9$

$x = 3 \longrightarrow 9 + y = 27$

$y = 18$

$P(0) = (0)(27) = 0$

$P(3) = (3)(18) = 54 \leftarrow$

$P(3\sqrt{3}) = (3\sqrt{3})(0) = 0$

(18)

$$F = \frac{v}{22+0.02v^2}$$

$$F' = \frac{(22+0.02v^2)(1)-(v)(0.04v)}{(22+0.02v^2)^2}$$

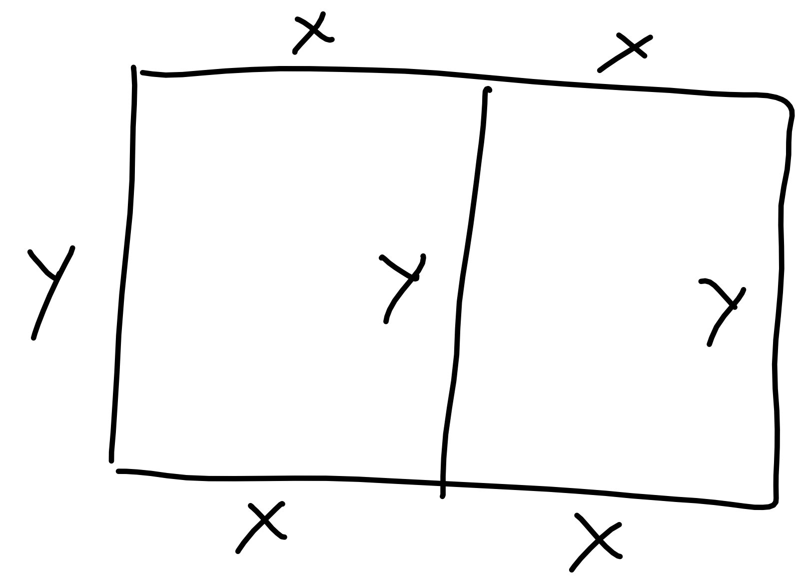
$$= \frac{22+0.02v^2-0.04v^2}{(22+0.02v^2)^2} = \frac{22-0.02v^2}{(22+0.02v^2)^2} = 0$$

$$0.02v^2 = 22$$

$$\sqrt{v^2} = \sqrt{1100}$$

$$v = 33.166 \text{ mi/hr}$$

(20)



$$P = 4x+3y = 200$$

$$4x = 200-3y$$

$$x = 50-\tfrac{3}{4}y$$

$$A = 2xy$$

$$A = 2\left(50-\tfrac{3}{4}y\right)y$$

$$A = \left(100-\tfrac{3}{2}y\right)y = 100y-\tfrac{3}{2}y^2$$

$$A' = 100-3y = 0$$

$$y = \frac{100}{3}$$

$$x = 50-\frac{\cancel{3}}{4}\left(\frac{100}{\cancel{3}}\right) = 50-25 = 25$$

EX4 → Which points on the graph of $y = 9 - x^2$ are closest to $(0,3)$?

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, $(x, y), (0,3)$ $\quad x^2 = 9 - y$

$d = \sqrt{(x-0)^2 + (y-3)^2} = \sqrt{x^2 + (y-3)^2} = \sqrt{9 - y + (y-3)^2} = \sqrt{9 - y + y^2 - 6y + 9}$

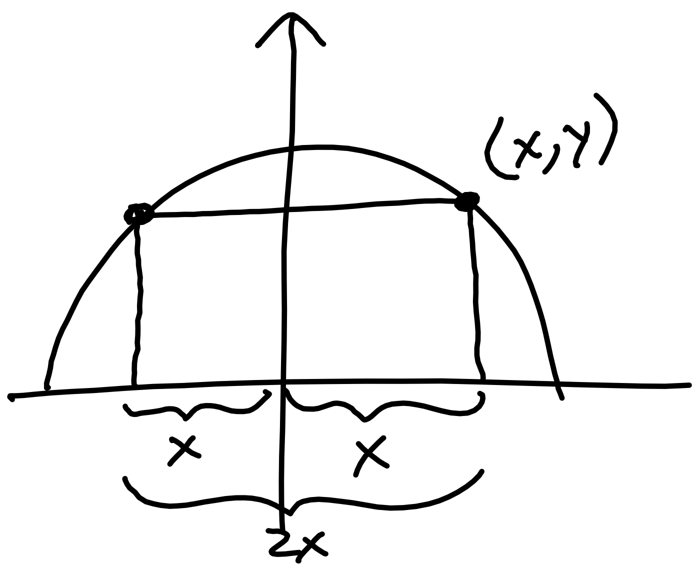
$d = \sqrt{y^2 - 7y + 18} \Rightarrow D = y^2 - 7y + 18$ (distance smallest when D is smallest)

$D' = 2y - 7 = 0$

$y = \frac{7}{2}$ $\quad x^2 = 9 - \frac{7}{2} = \frac{11}{2}$

$x = \pm\sqrt{\frac{11}{2}} \Rightarrow \boxed{\left(\sqrt{\frac{11}{2}}, \frac{7}{2}\right), \left(-\sqrt{\frac{11}{2}}, \frac{7}{2}\right)}$

EX5 $\rightarrow$ A rectangle is bounded by x-axis and semi-circle w/ equation $y=\sqrt{64-x^2}$. What length should rectangle have so the area is maximized?



$$A=l\cdot w=(2x)(y)=(2x)\left(\sqrt{64-x^2}\right) \rightarrow (64-x^2)^{1/2}$$

$$A'=(2x)\left(\tfrac{1}{2}(64-x^2)^{-1/2}\cdot -2x\right)+\left(\sqrt{64-x^2}\right)(2)=\frac{-2x^2}{\sqrt{64-x^2}}+2\sqrt{64-x^2}=0$$

$$\sqrt{64-x^2}\cdot 2\sqrt{64-x^2}=\frac{2x^2}{\cancel{\sqrt{64-x^2}}}\cdot\cancel{\sqrt{64-x^2}}$$

$$2(64-x^2)=2x^2 \Rightarrow 128-2x^2=2x^2 \Rightarrow 128=4x^2$$

$$x^2=32$$

$$x=4\sqrt{2} \Rightarrow l=2(4\sqrt{2})=8\sqrt{2}$$

HW: p. 223 → 13-17, 24, 25, 33

(14) $f(x)=\sqrt{x-8}$, $(2,0)$, (x,y) / $y^2=x-8$

$d=\sqrt{(x-2)^2+(y-0)^2}=\sqrt{(x-2)^2+y^2}$

$d=\sqrt{x^2-4x+4+x-8}=\sqrt{x^2-3x-4}$

$D=x^2-3x-4$

$D'=2x-3=0$

$x=\frac{3}{2}$

$(8,0)$

(16) $f(x)=(x+1)^2$, $(5,3)$, (x,y)

$d=\sqrt{(x-5)^2+(y-3)^2}=\sqrt{(x-5)^2+((x+1)^2-3)^2}$

$d=\sqrt{x^2-10x+25+(x^2+2x-2)^2}$

$d=\sqrt{x^2-10x+25+x^4+2x^3-2x^2+2x^3+4x^2-4x-2x^2-4x+4}$

$d=\sqrt{x^4+4x^3+x^2-18x+29}$

$D=x^4+4x^3+x^2-18x+29 \Rightarrow D'=4x^3+12x^2+2x-18=0$

$2(2x^3+6x^2+x-9)=0$

$x=1$

$f(1)=(1+1)^2=4 \Rightarrow (1,4)$

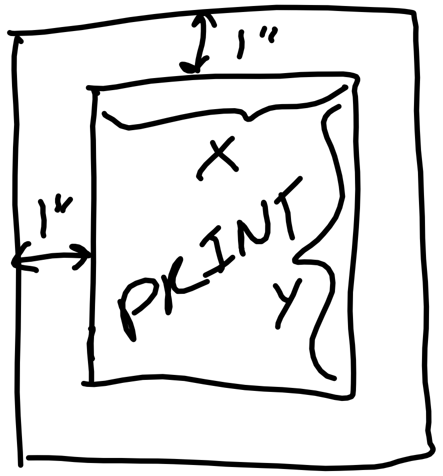
17 $\frac{dQ}{dx} = kx(Q_0 - x) = kQ_0x - kx^2$

$\frac{d^2Q}{dx^2} = kQ_0 - 2kx = 0$

$2\cancel{k}x = \cancel{k}Q_0$

$x = \frac{Q_0}{2}$

EX6 → A rectangular page contains 36 square inches of print w/ margins of 1 inch on either side. What is the minimum amount of paper used?



$$xy = 36 \Rightarrow x = \frac{36}{y}, \quad [1, 36]$$

$$A_o = (x+2)(y+2) = \left(\frac{36}{y} + 2\right)(y+2)$$

$$= 36 + \frac{72}{y} + 2y + 4 = \frac{72}{y} + 2y + 40$$

$$A_o' = -\frac{72}{y^2} + 2 = 0$$

$$\frac{72}{y^2} = 2$$

$$2y^2 = 72$$

$$y^2 = 36 \Rightarrow y = 6$$

$$A_o(1) = (1+2)(36+2) = (3)(38) = 114$$

$$A_o(6) = (6+2)(6+2) = (8)(8) = 64 \leftarrow$$

$$A_o(36) = (36+2)(1+2) = (38)(3) = 114$$

HW: p. 224 → 23, 27, 29, 30, 39