Related Rates

- Measuring the rate of change of two related variables w/ respect to time
  - ↳ implicit differentiation is used in same manner

EX1 → $4x^2 - 3y^2 = 1$, $\frac{dx}{dt} = -3$, $(1,1)$, $\frac{dy}{dt} = ?$

$$8x \cdot \frac{dx}{dt} - 6y \cdot \frac{dy}{dt} = 0$$

$$8(1) \cdot (-3) - 6(1) \cdot \frac{dy}{dt} = 0$$

$$-24 - 6\frac{dy}{dt} = 0$$

$$-6\frac{dy}{dt} = 24$$

$$\frac{dy}{dt} = -4$$

EX2 → $5xy = 10$, $(1,2)$, $\frac{dx}{dt} = 7$, $\frac{dy}{dt} = ?$

$$5x \cdot \frac{dy}{dt} + y \cdot 5 \cdot \frac{dx}{dt} = 0$$

$$5x\frac{dy}{dt} + 5y\frac{dx}{dt} = 0$$

$$5(1)\frac{dy}{dt} + 5(2)(7) = 0$$

$$5\frac{dy}{dt} = -70$$

$$\frac{dy}{dt} = -14$$

EX3 → What rate is the area of a circle changing when the radius is at 5 ft and is increasing at a rate of 2 ft/s ?

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi (5\,ft)(2\,ft/s)$$

$$\frac{dA}{dt} = 20\pi\ ft^2/s$$

EX4 → What rate is the area of a rectangle changing when the length is 4 ft, the width is 3 ft and both are changing at a rate of 7 ft/min?

$$A = l \cdot w$$

$$\frac{dA}{dt} = l \cdot \frac{dw}{dt} + w \cdot \frac{dl}{dt}$$

$$\frac{dA}{dt} = (4\,ft)(7\,ft/min) + (3\,ft)(7\,ft/min)$$

$$\frac{dA}{dt} = 49\ ft^2/min$$

HW: p. 154 → 1-8, 15, 18

20) $V = s^3$

$$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$$

A) $\frac{dV}{dt} = 3(1\text{cm})^2 (3 \text{ cm/s})$

$\frac{dV}{dt} = 9 \text{ cm}^3/\text{s}$

B) $\frac{dV}{dt} = 3(10\text{ cm})^2 (3 \text{ cm/s})$

$\frac{dV}{dt} = 900 \text{ cm}^3/\text{s}$

21) $A = 6s^2$

$$\frac{dA}{dt} = 12s \cdot \frac{ds}{dt}$$

A) $\frac{dA}{dt} = 12(1\text{ cm})(3 \text{ cm/s})$

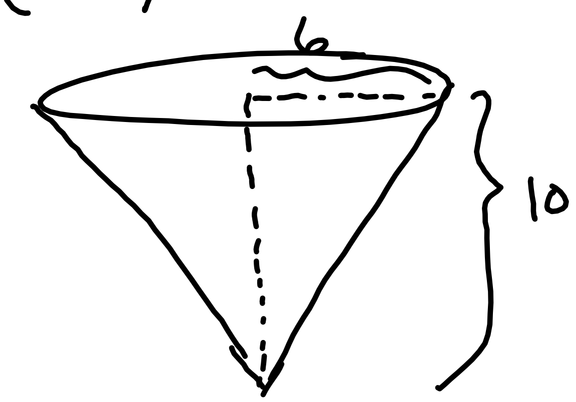
$\frac{dA}{dt} = 36 \text{ cm}^2/\text{s}$

B) $\frac{dA}{dt} = 12(10\text{ cm})(3 \text{ cm/s})$

$\frac{dA}{dt} = 360 \text{ cm}^2/\text{s}$

- When multiple variables are involved (especially w/ 3-D shapes), you will often have to express one in terms of the other

EX5 → Water is being pumped into a conical tank at a rate of 2 ft³/min. At what rate is the height of the water changing when the depth of the water is 8ft? (Height of cone = 10 ft, radius of cone = 6 ft)



$$\frac{h}{r} = \frac{10}{6} = \frac{5}{3}$$

$$5r = 3h$$

$$r = \frac{3}{5}h$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 \cdot h = \frac{3}{25}\pi h^3$$

$$\frac{dV}{dt} = \frac{9}{25}\pi h^2 \cdot \frac{dh}{dt}$$

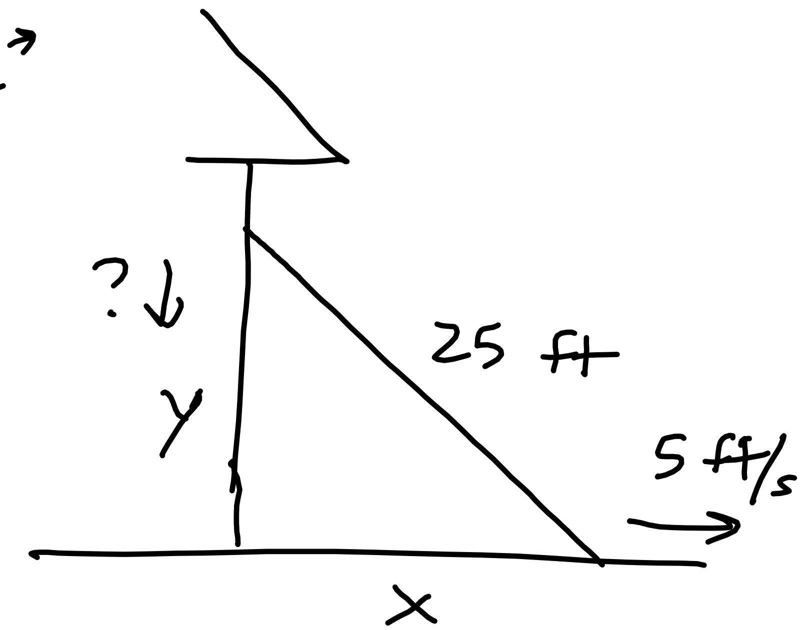
$$2\ \text{ft}^3/\text{min} = \frac{9}{25}\pi(8\ \text{ft})^2 \cdot \frac{dh}{dt}$$

$$2\ \frac{\text{ft}^3}{\text{min}} = \frac{576}{25}\pi\ \text{ft}^2 \cdot \frac{dh}{dt}$$

$$\frac{25}{288\pi}\ \text{ft}/\text{min} = \frac{dh}{dt}$$

- Distance / Pythagorean Theorem

EX →



$x = 7$

$y = 24$

$$x^2 + y^2 = 25^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(7\text{ft})(5\text{ ft/s}) + 2(24\text{ft})\frac{dy}{dt} = 0$$

$$48\text{ ft}\cdot\frac{dy}{dt} = -70\text{ ft}^2/\text{s}$$

$$\frac{dy}{dt} = \frac{-70}{48}\text{ ft/s} = -\frac{35}{24}\text{ ft/s}$$

EX → Distance b/w origin & $y = 1 + x^2$, $\frac{dx}{dt} = 1$ @ $x = 2$

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + (1+x^2)^2}$$

$$d = \sqrt{x^2 + 1 + 2x^2 + x^4} = \sqrt{1 + 3x^2 + x^4} = (1 + 3x^2 + x^4)^{1/2}$$

$$\frac{dd}{dt} = \frac{1}{2}(1 + 3x^2 + x^4)^{-1/2}\cdot\left(6x\frac{dx}{dt} + 4x^3\frac{dx}{dt}\right)$$

$$\frac{dd}{dt} = (1 + 3x^2 + x^4)^{-1/2}\left(3x\frac{dx}{dt} + 2x^3\frac{dx}{dt}\right)$$

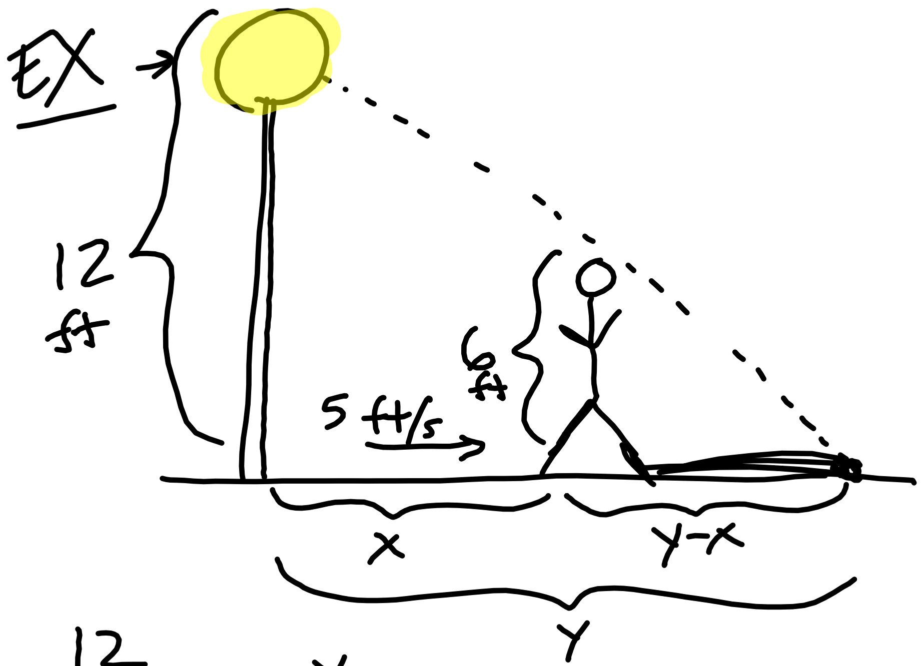
$$\frac{dd}{dt} = (1 + 12 + 16)^{-1/2}((6)(1) + (16)(1))$$

$$\frac{dd}{dt} = \frac{1}{\sqrt{29}}\cdot 22 = \frac{22}{\sqrt{29}}$$

HW: p. 154 → 22-24, 27A, 28, 31, 32

– Man + Lightpost

EX



$$\frac{12}{6} = \frac{y}{x}$$

$$2 = \frac{y}{x}$$

$$y = 2x$$

A) Rate of tip of shadow?

$$\frac{dy}{dt} = 2\frac{dx}{dt}$$

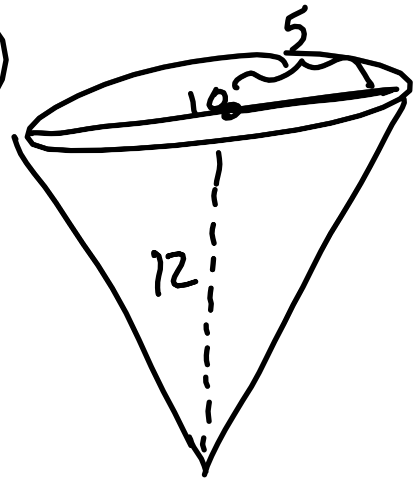
$$\frac{dy}{dt} = 2(5\text{ ft/s}) = 10\text{ ft/s}$$

B) Length of shadow

$$\frac{dy}{dt} - \frac{dx}{dt} = 10\text{ ft/s} - 5\text{ ft/s}$$

$$= 5\text{ ft/s}$$

24



$$\frac{5}{12} = \frac{r}{h}$$

$$r = \frac{5}{12}h$$

$$\frac{dV}{dt} = 10 \text{ ft}^3/\text{min}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h$$

$$V = \frac{25}{432}\pi h^3$$

$$\frac{dV}{dt} = \frac{25}{144}\pi h^2 \frac{dh}{dt}$$

$$10 \text{ ft}^3/\text{min} = \frac{25}{144}\pi (8 \text{ ft})^2 \frac{dh}{dt}$$

$$10 \text{ ft}^3/\text{min} = \frac{100\pi}{9} \text{ft}^2 \frac{dh}{dt}$$

$$\frac{90}{100\pi} \text{ ft}/\text{min} = \frac{dh}{dt} = \frac{9}{10\pi} \text{ ft}/\text{min}$$

27

$$r^2 + x^2 = 25^2$$

A) $2r\frac{dr}{dt} + 2x\frac{dx}{dt} = 0$

A) $x = 7 \Rightarrow r = 24$

$$2(24)\frac{dr}{dt} + 2(7)(2) = 0$$

⋮

B) $x = 15 \Rightarrow r = 20$

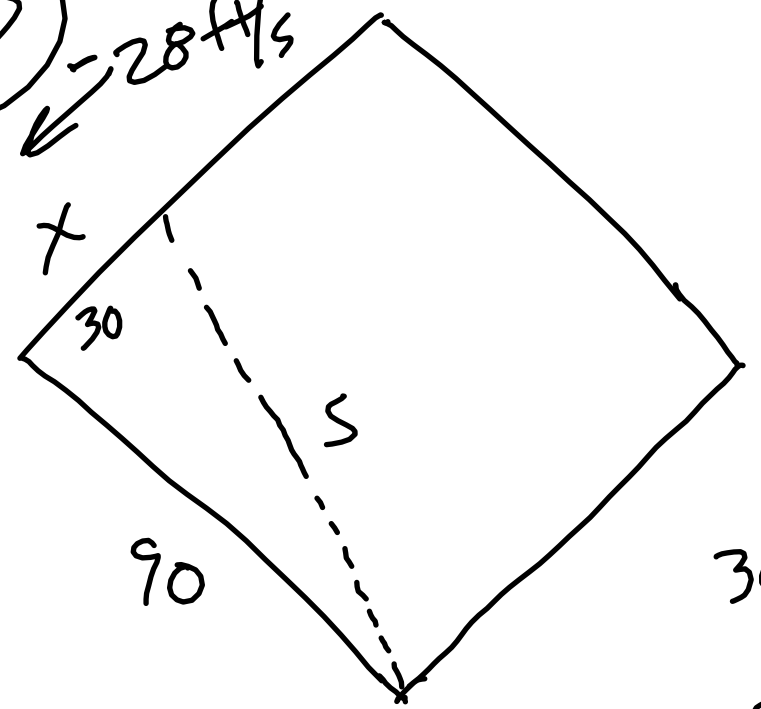
$$2(20)\frac{dr}{dt} + 2(15)(2) = 0$$

⋮

C) $x = 24 \Rightarrow r = 7$

HW: p. 154 → 12, 13, 33–35

33



$$30^2 + 90^2 = s^2$$

$$s = \sqrt{900 + 8100} = \sqrt{9000}$$

$$s = 30\sqrt{10}$$

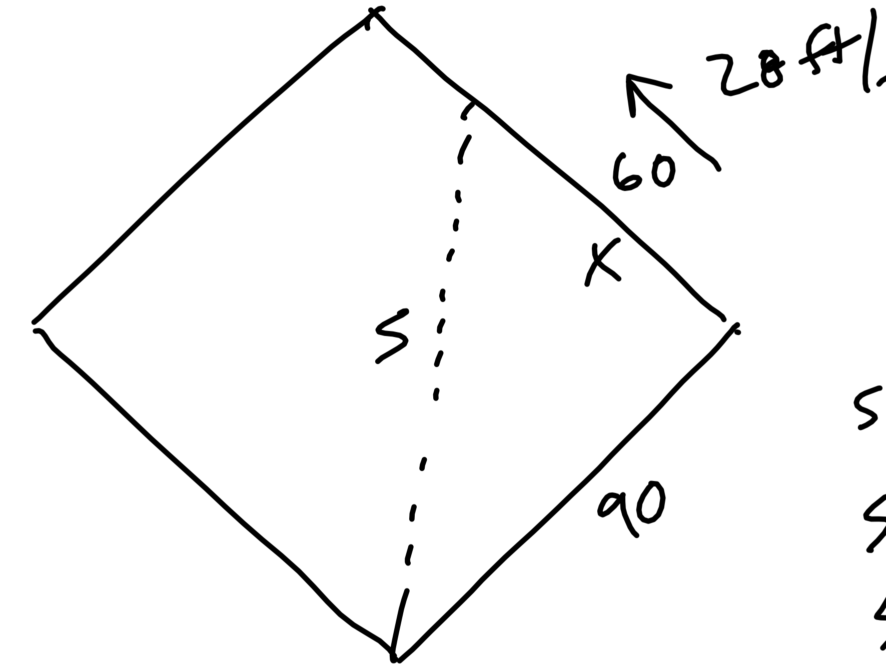
$$x^2 + 90^2 = s^2$$

$$\cancel{2}x \cdot \frac{dx}{dt} = \cancel{2}s \frac{ds}{dt}$$

$$(30\,\cancel{ft})(28\ ft/s) = (30\sqrt{10}\ \cancel{ft})\left(\frac{ds}{dt}\right)$$

$$\frac{ds}{dt} = \frac{(\cancel{30})(28\ ft/s)}{\cancel{30}\sqrt{10}} = \boxed{8.854\ ft/s}$$

34



$$s^2 = 60^2 + 90^2$$

$$s = \sqrt{3600 + 8100}$$

$$s = \sqrt{11700}$$

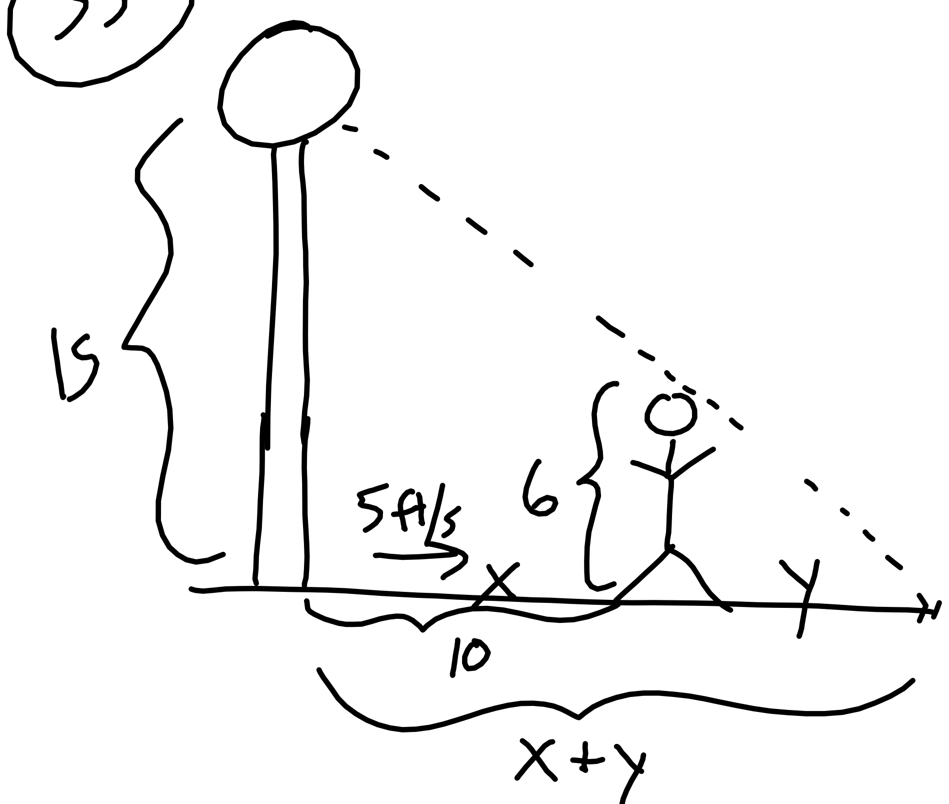
$$s^2 = x^2 + 90^2$$

$$\cancel{2}s \frac{ds}{dt} = \cancel{2}x \frac{dx}{dt}$$

$$(30\sqrt{13}\ \cancel{ft})\left(\frac{ds}{dt}\right) = (60\ \cancel{ft})(28\ ft/s)$$

$$\frac{ds}{dt} = \frac{(60)(28\ ft/s)}{30\sqrt{13}} = \boxed{15.531\ ft/s}$$

35



$$\frac{15}{6} = \frac{x+y}{y}$$

$$15y = 6x + 6y$$

$$9y = 6x$$

$$y = \frac{2}{3}x$$

$$\frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{2}{3}\left(5 \text{ ft/s}\right)$$

$$\frac{dy}{dt} = \frac{10}{3} \text{ ft/s}$$

$$\frac{d(x+y)}{dt}$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 5 \text{ ft/s} + \frac{10}{3} \text{ ft/s}$$

$$= \frac{25}{3} \text{ ft/s}$$