# U-Substitution

- A quick reminder of the chain rule: $\frac{d}{dx}\left[F(g(x))\right] = F'(g(x)) \cdot g'(x)$

When integrating, we often look for instances of the chain rule

→ Let $g$ be a function whose range is an interval $I$, & let $f$ be a function that is continuous on $I$. If $g$ is differentiable on its domain & $F$ is an antiderivative of $f$ on $I$, then

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$$

If $u = g(x)$, then $du = g'(x)\,dx$, and $\cancel{dx}\,\frac{du}{\cancel{dx}} = g'(x) \cdot dx$

$$\int f(u)\,du = F(u) + C$$

- Hints to performing U-substitution:
  - → Identify something that could be the derivative of something else
  - → You can multiply or divide du by something to make everything work
  - → You can manipulate u to match what is in the integral
  - ⊛ → Often, u will be inside of something else
  - → When performing U-substitution w/ a definite integral, pay close attention to what the limits of integration are expressed in

EX1 → $\int \underbrace{(x^2+1)}_{u}{}^2 \cdot \underbrace{2x\,dx}_{du}$

$u = x^2+1$

$du = 2x\,dx$

$= \int u^2 \cdot du$

$= \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}(x^2+1)^3 + C}$

EX2 → $\int 5 \sin 5x \, dx$

$u = 5x$

$du = 5\,dx$

$= \int \sin \underbrace{5x}_{u} \cdot \underbrace{5\,dx}_{du}$

$= \int \sin u \cdot du$

$= -\cos u + C = \boxed{-\cos 5x + C}$

EX3 → $\int 6 \sec^2 6x \, dx$

$u = 6x$

$du = 6\,dx$

$= \int \sec^2 u \, du$

$= \tan u + C = \boxed{\tan 6x + C}$

EX4 → $\int (x^2+1)^2 \cdot 4x\, dx$

$u = x^2+1$

$du = 2x\,dx \Rightarrow 2 \cdot du = 4x\,dx$

$= \int u^2 \cdot 2\,du = \int 2u^2\,du$

$= \frac{2}{3}u^3 + C = \boxed{\frac{2}{3}(x^2+1)^3 + C}$

EX5 → $\int (x^4+9)^4 \cdot 9x^3\, dx$

$u = x^4+9$

$du = 4x^3\,dx \Rightarrow \frac{9}{4}\,du = 9x^3\,dx$

$= \int u^4 \cdot \frac{9}{4}\,du = \int \frac{9}{4}u^4\,du$

$= \frac{9}{20}u^5 + C = \boxed{\frac{9}{20}(x^4+9)^5 + C}$

HW: p. 304 → 1-33 odd

EX6 $\rightarrow \int x\sqrt{2x-1}\,dx$

$u=2x-1 \Rightarrow \frac{u+1}{2}=x$

$du=2dx \Rightarrow \frac{1}{2}du=dx$

$$=\int \frac{u+1}{2}\cdot\sqrt{u}\cdot\frac{1}{2}du=\int \frac{1}{4}\left(u^{3/2}+u^{1/2}\right)du=\int \frac{1}{4}u^{3/2}+\frac{1}{4}u^{1/2}du$$

$$=\frac{1}{10}u^{5/2}+\frac{1}{6}u^{3/2}+C=\boxed{\frac{1}{10}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}+C}$$

EX7 $\rightarrow \int x\sqrt{3x-4}\,dx$

$u=3x-4 \Rightarrow \frac{u+4}{3}=x$

$du=3dx \Rightarrow \frac{1}{3}du=dx$

$$=\int \frac{u+4}{3}\sqrt{u}\cdot\frac{1}{3}du=\int \frac{1}{9}u^{3/2}+\frac{1}{9}\cdot 4\sqrt{u}\,du=\int \frac{1}{9}u^{3/2}+\frac{4}{9}u^{1/2}du$$

$$=\frac{2}{45}u^{5/2}+\frac{8}{27}u^{3/2}+C=\boxed{\frac{2}{45}(3x-4)^{5/2}+\frac{8}{27}(3x-4)^{3/2}+C}$$

HW: p. 305 → 43-67 odd

51) $\int \tan^4 x \sec^2 x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int u^4 \, du$

$= \frac{1}{5}u^5 + C = \boxed{\frac{1}{5}\tan^5 x + C}$

53) $\int \frac{\csc^2 x}{\cot^3 x} \, dx$

$u = \cot x$
$du = -\csc^2 x \, dx \Rightarrow -du = \csc^2 x \, dx$

$= \int \frac{-1}{u^3} \, du = \int -u^{-3} \, du$

$= \frac{1}{2}u^{-2} + C = \frac{1}{2}(\cot x)^{-2} + C = \boxed{\frac{1}{2\cot^2 x} + C}$

55) $\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$

$\downarrow$

$1 + \cot^2 x = \csc^2 x$

$\cot^2 x = \csc^2 x - 1$

$\int \cot^2 x \, dx = \int (\csc^2 x - 1) \, dx$

$= \boxed{-\cot x - x + C}$

65
67

(65) $\int x^2\sqrt{1-x}\,dx$

$u = 1-x \Rightarrow x = 1-u$

$du = -dx \Rightarrow -du = dx$

$= -\int (1-u)^2\sqrt{u}\,du = -\int (1-2u+u^2)\sqrt{u}\,du = -\int u^{1/2} - 2u^{3/2} + u^{5/2}\,du$

$= \int -u^{1/2} + 2u^{3/2} - u^{5/2}\,du = -\frac{2}{3}u^{3/2} + \frac{4}{5}u^{5/2} - \frac{2}{7}u^{7/2} + C = \boxed{-\frac{2}{3}(1-x)^{3/2} + \frac{4}{5}(1-x)^{5/2} - \frac{2}{7}(1-x)^{7/2} + C}$

(67) $\int \frac{x^2-1}{\sqrt{2x-1}}\,dx$

$u = 2x-1 \Rightarrow \frac{u+1}{2} = x$

$du = 2\,dx \Rightarrow \frac{1}{2}du = dx$

$= \frac{1}{2}\int \frac{\left(\frac{u+1}{2}\right)^2 - 1}{\sqrt{u}}\,du = \frac{1}{2}\int \frac{\frac{u^2+2u+1}{4} - 1}{\sqrt{u}}\,du = \frac{1}{2}\int \frac{\frac{u^2+2u+1}{4} - \frac{4}{4}}{\sqrt{u}}\,du = \frac{1}{2}\int \frac{\frac{u^2+2u-3}{4}}{\sqrt{u}}\,du$

$= \frac{1}{2}\int \frac{u^2+2u-3}{4\sqrt{u}}\,du = \frac{1}{2}\int \frac{1}{4}\left(\frac{u^2+2u-3}{\sqrt{u}}\right)du = \frac{1}{8}\int u^{3/2} + 2u^{1/2} - 3u^{-1/2}\,du = \int \frac{1}{8}u^{3/2} + \frac{1}{4}u^{1/2} - \frac{3}{8}u^{-1/2}\,du$

$= \frac{1}{20}u^{5/2} + \frac{1}{6}u^{3/2} - \frac{3}{4}u^{1/2} + C = \boxed{\frac{1}{20}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} - \frac{3}{4}(2x-1)^{1/2} + C}$

**EX 8** → $\int_2^4 (x^3+1)^3 \cdot 3x^2\,dx$

$u = x^3+1$

$du = 3x^2\,dx$

$= \int_{x=2}^{x=4} u^3\,du$

OR

$\int_{u=9}^{u=65} u^3\,du$

$u = 2^3+1 = 9$

$u = 4^3+1 = 65$

$= \frac{1}{4}u^4 \Big|_{x=2}^{x=4}$

$= \frac{1}{4}u^4 \Big|_{u=9}^{u=65}$

$= \frac{1}{4}(x^3+1)^4 \Big|_2^4$

$= \frac{1}{4}(65)^4 - \frac{1}{4}(9)^4$

$= 4461016$

$= \left[\frac{1}{4}(65)^4\right] - \left[\frac{1}{4}(9)^4\right]$

$= 4461016$

HW: p. 305 → 71-85 odd, 109, 110, 113

(77) $$\int_1^9 \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx$$

$u = 1+\sqrt{x}$

$du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$

$$= \int_{x=1}^{x=9} \frac{1}{u^2}\cdot 2\,du = \int_{x=1}^{x=9} \frac{2}{u^2}\,du = \int_{x=1}^{x=9} 2u^{-2}\,du$$

$$= -2u^{-1}\Big|_{x=1}^{x=9} = -2(1+\sqrt{x})^{-1}\Big|_1^9 = \left[\frac{-2}{1+\sqrt{9}}\right] - \left[\frac{-2}{1+\sqrt{1}}\right] = \frac{-2}{4} + \frac{2}{2} = -\frac{1}{2}+1 = \frac{1}{2}$$

(83) $\frac{dy}{dx} = 18x^2(2x^3+1)^2$, $(0,4)$

$\int dy = \int 18x^2(2x^3+1)^2\,dx$

$u = 2x^3+1$

$du = 6x^2\,dx \Rightarrow 3du = 18x^2dx$

$y = \int 3u^2\,du$

$y = u^3 + C \Rightarrow y = (2x^3+1)^3 + C$

$4 = (0+1)^3 + C \Rightarrow C = 3 \Rightarrow \boxed{y = (2x^3+1)^3 + 3}$

113

A) $R = 3.121 + 2.399\sin(0.524t + 1.377)$

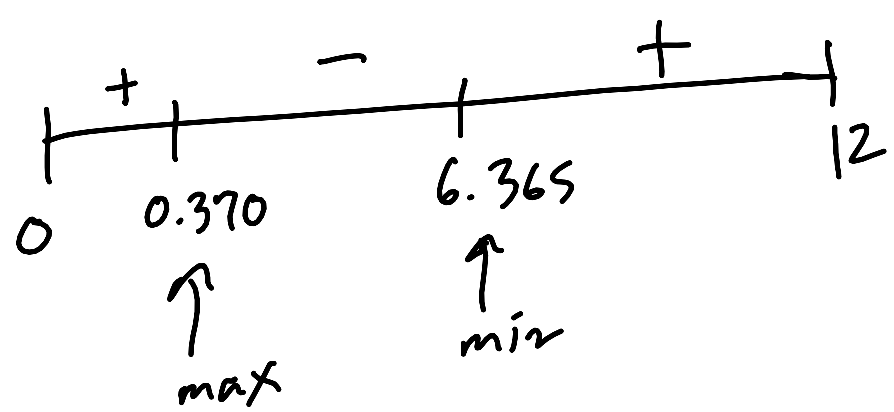
$R' = 2.399\cos(0.524t + 1.377) \cdot 0.524$

$R' = 1.2571\cos(0.524t + 1.377) = 0$

$t = 0.370,\ 6.365$

max @ $t = 0.370$

min @ $t = 6.365$




B) $\int_0^{12} R(t)\,dt = 37.474$ in.

C) $\frac{1}{3}\int_9^{12} R(t)\,dt = \frac{1}{3}(12.999) = 4.333$ in.